Co-Holding with Closed-End Loans*

Jared Buck†

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Abstract

Households often hold liquid assets while carrying debt at a higher interest rate. Rational explanations of this co-holding puzzle are built for credit cards, yet the Survey of Consumer Finances shows that co-holding with mortgages and installment loans is at least as costly. In 2022 the median mortgage co-holder gave up $1,055 a year in excess interest, compared with $393 for credit cards. This paper develops a three-period theory of co-holding with closed-end loans. Because repaid principal cannot be borrowed back, paying down debt early reduces the debt a household can carry later, so liquid assets and debt coexist when households use loans to tap future resources and to self-insure against expense shocks. I derive the conditions under which co-holding is optimal and show that it survives when households also have access to a line of credit and to a high-return illiquid asset. Net worth is a sufficient statistic for welfare in the deterministic model but not once expense risk is introduced. Regressions on the Survey of Consumer Finances are consistent with the model's predictions.

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†Department of Economics, Concordia University. Email: jared-wb@hotmail.com.

1 Introduction

In the Survey of Consumer Finances (SCF) 2022, nearly 20% of households are indebted while simultaneously holding liquid savings in excess of three months of income. For some liabilities, such as mortgage debt originated during historic low interest rates, this may not be puzzling. In these cases, the interest cost of debt may be low enough that holding liquid savings is not obviously dominated by repayment. However, the coexistence of substantial liquid savings and costly consumer debt is harder to reconcile. For example, around 8% of households carry revolving credit card debt despite holding large liquid buffers, hence the *credit card debt puzzle*. Installment debt presents a related, though less obvious, version of the same issue. On a simple rate-of-return basis, paying down installment debt should dominate holding liquid assets, yet households still choose to co-hold.

The mechanism developed in this paper fits most directly in the rational co-holding literature (Telyukova 2013; Fulford 2015; Druedahl and Jørgensen 2018). In particular, Fulford (2015) and Druedahl and Jørgensen (2018) both introduce models where co-holding is rational because households are uncertain about future access to credit. However, since this paper studies installment debt, households don't have access to an open-end borrowing limit, they instead receive a fixed principal and pay it back over time. Because there is no uncertainty about future access to borrowing, in this paper, households choose a fixed loan at origination, and it is this lock-in effect that leads to co-holding. To illustrate, suppose an individual is considering purchasing a car and they have two options: they can pay with cash, or they can keep their cash and take out a loan. If they pay for it directly with cash, they will certainly save on interest payments, however, that may leave them without enough liquidity to pay for upcoming expenses. Therefore, in order to simultaneously purchase the vehicle and meet short-term liquidity needs, the household would need to finance the car through an installment loan.

The model also relates to the hand-to-mouth (HtM) literature (Kaplan and Violante 2014; Kaplan, Violante, and Weidner 2014; Aguiar, Bils, and Boar 2025). While HtM typically focuses on the interaction of liquid and illiquid assets, it largely abstracts from the role of debt structure. This paper connects the HtM framework to the co-holding puzzle by showing the role that closed-end loans play in lowering liquidity constraints. In particular, households who present themselves as wealthy HtM (WHtM) in Kaplan and Violante (2014) face the same tradeoff as co-holders, except they fall on opposite sides of the decision: for the co-holder, liquidity needs are sufficient to justify paying a premium to tap illiquid wealth. For the HtM, they may have liquidity needs, but if the cost of accessing that liquidity is too rich they may elect to have a larger net future payoff.

This paper is connected to the work on models that disaggregate household portfolios (Campbell and Cocco 2003; Nakajima and Telyukova 2017, 2020). To generate co-holding, this model has households simultaneously choosing liquid assets and debt contracts, given time specific endowments. In an extension to the model, I introduce a high return illiquid asset that can be tapped early at a cost. This type of asset resembles tax advantaged retirement savings in the US like IRA and 401(k) accounts. Focusing on portfolio choices reveals something about how net asset models (where assets take on positive or negative positions) may actually miss important distortions from policies: when tax policies to one portfolio object change, it doesn't just affect the net position, it also affects the distribution of the portfolio which is part of the welfare effect.

The baseline model makes two testable predictions. First, households with higher future endowments should both be more likely to co-hold and co-hold more at the intensive margin. This is because future endowments act like an illiquid asset that cannot be tapped early. If current endowments are relatively small, then the household has an incentive to take out a loan to smooth consumption, indirectly tapping their future endowment. Second, increased expense risks should also increase co-holding. If between the time the loan is originated and the illiquid endowment is accessible, households may face larger expenses, taking on more debt to ensure liquidity for bad expense shocks is rational for the risk averse. If the mechanisms from the model are operating, these patterns should emerge from the data. To test the model's predictions, I run a set of reduced form regressions on the SCF data. As proxies, I use income for current endowments and both retirement savings and income in a normal year for future endowments. While these regressions do not establish causality, households with higher relative future endowment proxies are more likely to co-hold.

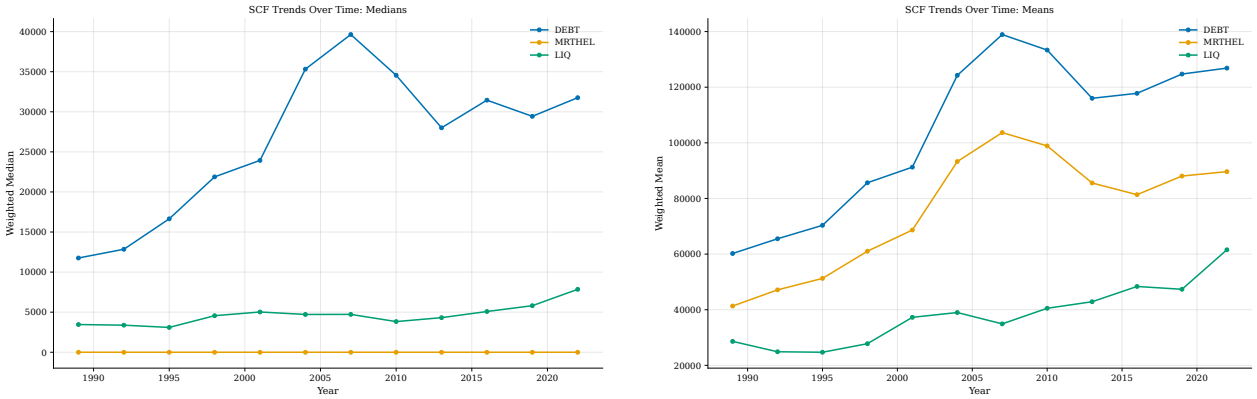
The remainder of the paper is organized as follows. Section 2 presents evidence of the co-holding puzzle from the data, and documents the stylized facts that motivate the model. Section 3 formally develops the baseline model and derives the conditions that lead to co-holding. Then, it compares implications and key comparative statics to that of a canonical net-worth model. Section 4 extends the baseline model adding expense shocks and testing how the mechanism holds up in environments that features high-return low-liquid assets, and consumer lines of credit. Section 5 presents the empirical findings of patterns the model predicts should exist in the data. Section 6 concludes.

2 Evidence From SCF

Data is sourced from the Survey of Consumer Finances (SCF). The SCF is the natural data source for this question because no other U.S. household survey provides comparably

detailed information on both sides of the household balance sheet (liquid assets, mortgages, installment debt, revolving credit, pensions, and income) together with survey design features intended to measure even the upper tail of the wealth distribution accurately. In this section, I use the SCF to document some stylized facts about household portfolios, taking the entire sample period from 1989-2022 of the summary extract to construct the observational variables. Since this section focuses primarily on observation and does not involve any inference, I rely on the sample weights to construct statistics rather than the SCF replicate weights, unless otherwise specified. I also refer to the summary extract to define measures of liquid assets, different types of debt, and income. The specific variables and methodologies used are defined in appendix A.

Figure 1 plots mean and median balances for various account types. The first thing to



(a) Weighted Medians

(b) Weighted Means

Figure 1: SCF Portfolio Trends

Note: Panels (a) and (b) plot weighted mean and median total debt (DEBT), debt secured by primary residence (MRTHEL), and liquid savings (LIQ) respectively using data from the SCF summary files. Definitions of DEBT, MRTHEL, and LIQ are provided in appendix A

make note of is that the median household has no mortgage balance. This is not a surprising result because less than half of households are carrying mortgages in a given year. Meanwhile, median liquid assets hover at around $5,000 as median debt increases from about $12,000 in 1989 to $32,000 in 2022, with a spike leading up to the housing crash. This trend points to US households choosing to hold less liquid buffers over time. Meanwhile, panel 1b shows the same trend for debt over the average where mortgages are overwhelmingly responsible for the vast majority of total debt (around 70%). At the same time, total liquid buffers have remained consistently at close to 40% of total debt (ignoring the spike in debt during the crisis). This is the type of pattern we'd expect to find if households are co-holding, though it is not a necessary condition.

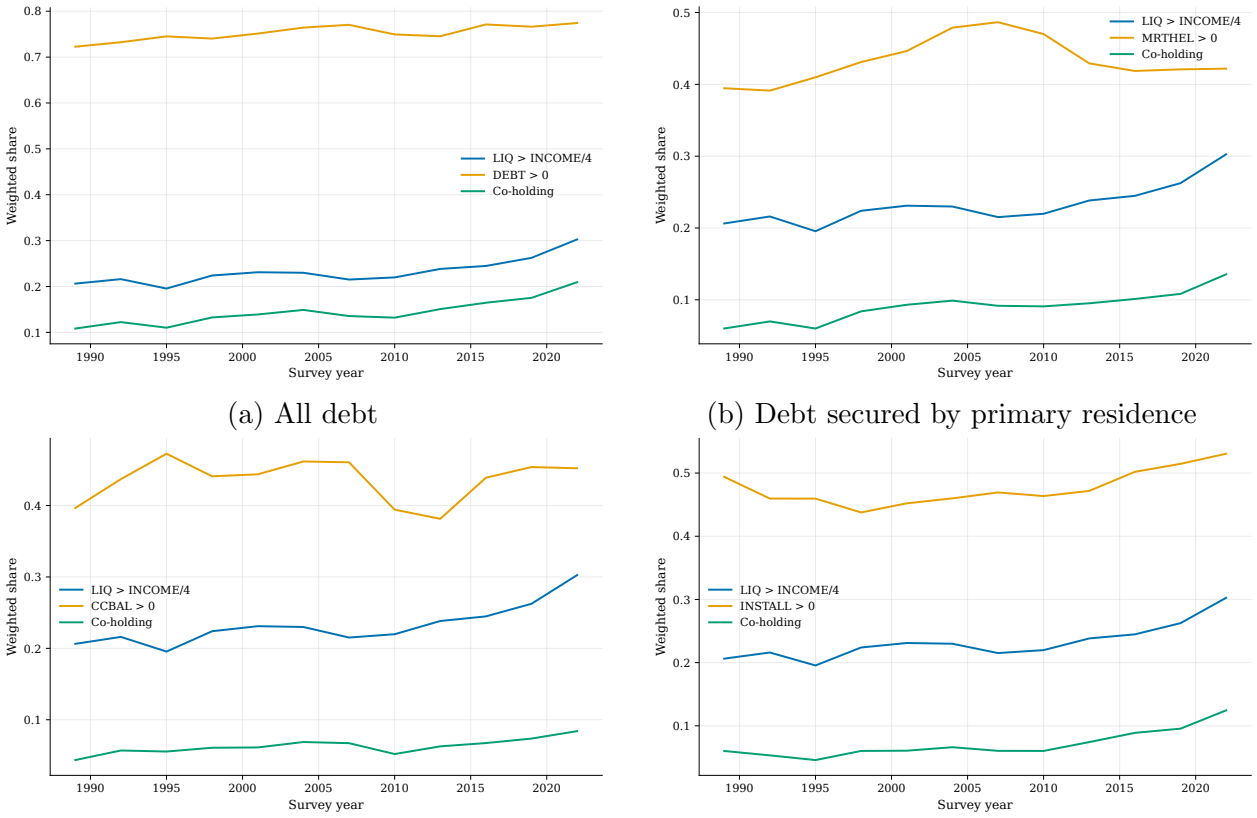
In general, I define households as co-holders if they simultaneously hold liquid assets and debt in their portfolio. In practice, because only very HtM households will have a balance of zero in their checking accounts, by this definition almost all indebted households would be considered co-holders. In reality, households with debt do not adjust continuously. Those living paycheck-to-paycheck, or attempting to repay debt as quickly as possible, generally make payments at discrete intervals (weekly or biweekly). Although these households are co-holders in a mechanical sense, they are not necessarily optimally choosing to retain both liquid assets and debt at the same time because income receipts, purchases, bill payments, and debt repayments occur at different dates[1]. For this reason, at the extensive margin, I say a household is a co-holder if they simultaneously hold debt and liquid assets in excess of 3 months worth of income[2]. This threshold is intended to exclude trivial transaction balances and to isolate households holding a substantial liquid buffer despite carrying debt.

Figure 2 shows how co-holding at the extensive margin evolves across time from 1989 to 2022. The figures also decompose both components of co-holding to show how both debt shares and sufficient liquidity buffers move. The following key facts are revealed. First, co-holding is not rare and appears to be increasing. About 20% of households in 2022 were co-holding with at least one type of debt, and nearly 10% of households were co-holding with credit card debt despite the high interest rates. Since savings rates on liquid accounts (checking and other transaction accounts) tend to be quite low and credit card rates commonly exceed 15% it is hard to reconcile this behaviour in standard models. Second, the phenomenon is not limited to credit card debt. Crucially, both the share of households with installment debt and debt secured by primary residence exceed 10% by 2022. Since both these types of loans tend to be more structured and quite sizeable, this implies that co-holding is not just a billing-cycle timing issue. Third, from panel 2b in regards to debt secured by primary household, co-holding and liquid buffers appear to follow the same paths. However, since debt grows significantly from around 1998 to 2007 relative to co-holding and liquid buffers, co-holding doesn't appear to merely be a consequence of more households taking on debt. If anything, since households holding liquid assets greater than three months income is relatively stable, households may be choosing to hold debt rather than depleting, or even to maintain liquidity reserves.

The data also reveals that co-holding is concentrated just before and around the time people start to enter retirement. Table 1 reports co-holding by age for the same four types of

1. Telyukova (2013) in particular studies this type of timing mechanism to explain the credit card debt puzzle.

2. This definition is much stricter than typical measures. However, since the literature has focused on credit card debt, I argue that for long-term loans like mortgages and other installment loans, this is a more reasonable way of filtering out short-term co-holders.



(a) All debt

(b) Debt secured by primary residence

(c) Credit card debt

(d) Installment debt (non-mortgage)

Figure 2: Co-holding by type of debt

Note: Each panel uses the SCF summary extract variables. INCOME and LIQ are the variables used for income and liquid assets respectively. Panel (a) uses all debt (DEBT), panel (b) uses all debt secured by primary residence (MRTHEL), panel (c) uses revolving credit card debt (CCBAL), and panel (d) uses non-mortgage installment debt (INSTALL). The green line shows the share of co-holders with liquid assets greater than 3 months income, the orange line shows the share of debtors, and the blue line shows the share of liquid buffers.

debt. Mortgage co-holding in particular appears to peak between ages 50 to 59 but still

Table 1: Co-holding by age group

Age group	Credit card debt	Mortgage debt	Installment debt	Total debt
20–29	4.7%	3.3%	7.6%	10.3%
30–39	4.9%	7.6%	6.9%	11.2%
40–49	6.0%	11.5%	7.6%	14.4%
50–59	7.6%	12.7%	8.3%	17.7%
60–69	8.0%	11.1%	7.7%	18.4%
70–79	7.7%	8.3%	5.7%	16.7%
80–89	5.0%	4.6%	3.4%	11.0%

Notes: Each cell reports the simple average across SCF survey waves of the wave-specific weighted share of households in the indicated age group that are co-holders. A household is a co-holder when it has positive debt in the relevant debt category and liquid assets greater than three months of current income, LIQ > INCOME/4. Wave-specific statistics use the SCF household sampling weights. Credit-card debt uses CCBAL, mortgage debt uses MRTHEL, installment debt uses INSTALL, and total debt uses DEBT. The sample is restricted to ages 20 through 89.

persists over 8% through ages 70-79. A companion paper, *Debt as Insurance*, studies the co-holding problem in the context of retirement in much more detail. More importantly, estimates from figure 2 and table 1 are quite different from the typical 30% share often found in the literature, but this is expected from the 3-month threshold. Table 2 computes co-holding at the extensive margin across the entire sample for different thresholds. Widening cutoffs to typical thresholds produces the consistent 30%-40% measures often cited in the literature (Telyukova 2013; Fulford 2015; Druedahl and Jørgensen 2018).

Despite the 3-month income threshold, the share of co-holders doesn't inform us about how much people actually hold in buffers. At the intensive margin I refer back to the broad definition of liquid balances greater than zero. This allows the analysis to focus on the magnitude of balance-sheet positions rather than on threshold crossing. Figure 3 plots the ratio of debt to liquid buffers among all co-holders (in this case those with positive debt and liquid assets). The plot is computed first by ranking each co-holder by their debt to liquid holdings. Each observation is binned to the nearest decile and the points in the plot represent the median within decile. The ratio tells us how many dollars of debt each decile median has per dollar of liquid assets. Anything below the dashed line implies more liquid holdings than debt. Since credit card balances tend to be lower than longer-term secured forms of debt, it shouldn't be a surprise that there are stronger buffers. What is surprising is that more than half of credit-card co-holders have enough liquid assets to actually pay off their entire balances. Likewise, over 10% of installment debt co-holders could also pay off their debts in full. Mortgages tend to be quite large in balance, usually into the 6-figure ranges, so even the

Table 2: Co-holding shares under alternative definitions

Co-holding definition	Credit card debt	Mortgage debt	Installment debt	Total debt
LIQ > 0, $D > 0$	42.4%	42.4%	45.0%	71.5%
LIQ $>$ \$100, $D > 0$	40.8%	41.4%	42.7%	68.3%
LIQ $>$ \$100, $D >$ \$100	39.6%	41.4%	42.6%	67.9%
LIQ $>$ \$500, $D > 0$	37.3%	39.1%	38.7%	62.5%
LIQ $>$ \$500, $D >$ \$500	31.8%	39.0%	37.9%	60.7%
LIQ $>$ INCOME/12, $D > 0$	16.6%	20.5%	17.9%	32.0%
LIQ $>$ INCOME/4, $D > 0$	6.1%	8.9%	6.9%	14.1%
LIQ $> D$, $D > 0$	23.6%	3.2%	11.9%	10.7%

Notes: Each cell is the simple average across SCF survey waves of the wave-specific weighted share of households satisfying the stated co-holding definition. Wave-specific shares use the SCF household sampling weights. In each column, D denotes the debt balance named by that column: CCBAL for credit-card balances, MRTHEL for debt secured by the primary residence, INSTALL for non-mortgage installment debt, and DEBT for total debt. For income-based definitions, households with nonpositive income do not satisfy the threshold. Dollar thresholds are applied in the same dollar units as the monetary variables supplied to the function. In the final row, LIQ is compared with the corresponding debt balance in that column.

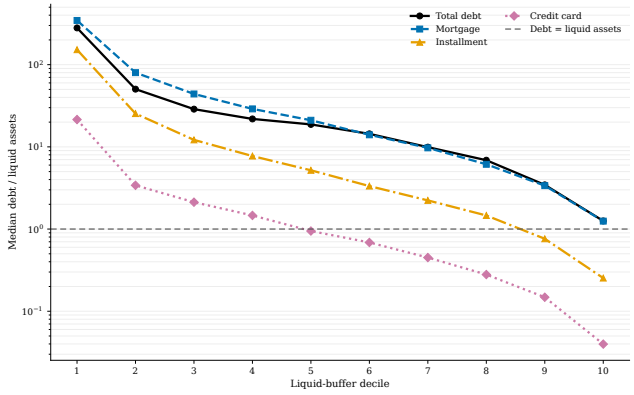


Figure 3: Debt to liquid asset ratio among co-holders

Note: This plot shows the decile medians of debt-to-liquid-asset ratios from the SCF summary extract variable across the four listed debt types. The dashed horizontal line is where the liquid asset balance is equal to debt. Each decile is computed by ranking all observations, binning them into deciles, and computing the weighted within-decile median.

highest decile of co-holders aren't quite able to pay it off, yet still, only the lowest decile has more that $100 in debt per $1 in savings. Regardless, this reveals that many co-holders are forgoing savings on debt repayment in the interest of carrying liquid buffers.

While figure 3 might reveal the extent to which co-holders can pay off their debts, unlike at the extensive margin, there is little indication to suggest how much the debt actually costs them. This is the materiality of co-holding, that is, the cost of the position. Assessing that cost is a bit of a challenge because the SCF does not report rates of return on liquid assets, and the rates on debt have to be pieced together from the raw data files. The question is simple: if people are willing to forgo paying off their debts, how much does this cost them? Substantively, the answer to this question is

$$\text{Carry} = (r_d - r_a)\min\{\text{LIQ}, \text{DEBT}\} \tag{1}$$

where Carry is the carrying cost of co-holding, r_d is the interest rate on debt, r_a is the interest rate on assets, LIQ is liquid asset holding, and DEBT is debt holdings.

DEBT is the easiest part to handle, simply condition co-holding on debt type (credit card, mortgage, or installment). However, interest rates on debt accounts vary at the household level as well, so these need to be inferred directly from the raw data. For each debt account reported by the household, the SCF asks them to report the interest rate they paid. In the summary extract, debt is aggregated into types. For example, MRTHEL includes up to three mortgage and three HELOC accounts secured by the primary residence. I use a balance-weighted average of the accounts within debt type to approximate the marginal rate paid on debt

$$r_{\text{DEBT},h} = \frac{\sum_j b_{j,h} \cdot r_{j,h}}{\sum_j b_{j,h}}. \tag{2}$$

where h indexes household, j indexes the debt component, b denotes the balance, and r denotes the interest rate. To ensure this works more as a lower bound on the implicit co-holding premium, if a rate is missing, the balance of the missing rate is dropped from the computation[3].

LIQ adds a complication in that households do not report returns on their savings accounts like they do for debt, so these rates need to be approximated using FRED based on the account types inside LIQ. To approximate returns, I use a 0% return on checking accounts, the National Rate: Savings (SNDR), and the National Rate: Money Market <100M (MMNDR) time series data to approximate returns on checking accounts, saving accounts, and money market accounts respectively. The marginal return on LIQ savings, r_{LIQ}, is then determined

3. for example if DEBT $= d_1 + d_2$ and the rate on d_1 is missing, then $\hat{d} = \text{DEBT} - d_1$ is the balance used for the computation.

as the weighted average

$$r_{\text{LIQ},h} = \frac{\sum_j a_{j,h} \cdot r_j}{\sum_j a_{j,h}} \tag{3}$$

where a refers to the specific asset balance. While rates may differ at the household level, since most of these rates are near 0%, approximation of the costs should have a small error.

Table 3 reports the estimated carry costs for co-holders with liquid assets in excess of one month income. Despite being more of an intensive margin measure, the one month threshold is imposed because if many households are short-term co-holders (only co-hold for days or weeks), including them in the analysis may actually overstate their costs (Gathergood and Olafsson 2026). The results support the hypothesis by Zinman (2007) that including

Table 3: Forgone net interest on co-held positions using 1-month threshold, 2022 SCF.

Debt type	Co-holding share	Median offset ($)	Median r_{debt}	Median carry ($/yr)	Carry band ($/yr)
Credit card	20.2%	2,800	16.0%	393	329–396
Installment	25.5%	13,000	4.9%	552	211–566
Mortgage / HELOC	25.2%	32,500	3.3%	1,055	193–1,088
Any debt	40.6%	20,490	6.0%	1,206	698–1,229

Note: A household is a co-holder of a given debt type if it holds that debt and liquid assets above one month of income. The offset is $\min(LIQ, DEBT_k)$, the part of the balance sheet that could be netted out. Carry is the offset times the spread between the household's debt rate and the return on its liquid bundle. Rates are nominal; the spread, and hence the carry, is invariant to inflation. The band spans three assumptions about the liquid return: the upper end sets it to zero, the lower end to the best liquid rate available in the survey year, and the point estimate weights component returns by each household's own portfolio composition. Debt rates are household-reported where the full public dataset supplies them. The carry is a lower bound on what these households are willing to pay for liquidity, not a measure of loss.

installment and mortgage debt may reveal higher premiums. Indeed, table 3 suggests that in 2022, the median household is willing to pay $1,055 to hold $32,500 in liquid despite carrying at least $32,500 in mortgage. The literature has focused on co-holding with credit cards because of the massive offset in interest, however, other forms of debt may be where households pay the biggest premiums. Table 4 reports the same statistics conditional on the three month asset buffer. The results reveal more about the tail of the co-holding distribution: co-holding share is roughly halved, but the costs for mortgagors is doubled.

The last question worth discussing concerns who is co-holding. Table 5 displays average household portfolios by co-holding type. Here, co-holders meet the extensive margin threshold, such that debtors with without meaningful liquid assets are restricted to those with not more than three months income as a buffer. Perhaps the most salient fact here is that co-holders appear both to have higher incomes and liquid assets. The implication being

Table 4: Forgone net interest on co-held positions using 3-month threshold, 2022 SCF.

Debt type	Co-holding share	Median offset ($)	Median r_{debt}	Median carry ($/yr)	Carry band ($/yr)
Credit card	8.3%	2,500	16.0%	375	314–378
Installment	12.2%	18,000	4.8%	757	267–780
Mortgage / HELOC	13.5%	59,120	3.2%	2,047	369–2,100
Any debt	20.7%	42,000	6.0%	2,462	1,431–2,520

Note: A household is a co-holder of a given debt type if it holds that debt and liquid assets above three months of income. The offset is $\min(LIQ, DEBT_k)$, the part of the balance sheet that could be netted out. Carry is the offset times the spread between the household's debt rate and the return on its liquid bundle. Rates are nominal; the spread, and hence the carry, is invariant to inflation. The band spans three assumptions about the liquid return: the upper end sets it to zero, the lower end to the best liquid rate available in the survey year, and the point estimate weights component returns by each household's own portfolio composition. Debt rates are household-reported where the full public dataset supplies them. The carry is a lower bound on what these households are willing to pay for liquidity, not a measure of loss.

Table 5: Summary statistics by co-holding status

	All households	Co-holders	Debt holders w/o meaningful liquidity	Liquid-asset holders no debt
Income	141,390	185,640	122,615	185,532
Liquid assets	61,551	145,786	10,921	257,792
Total debt	126,864	212,044	146,002	0
Mortgage debt	89,643	158,269	100,377	0
Installment debt	19,532	23,650	25,623	0
Age	52.4	53.1	48.8	65.4
Education	9.9	11.0	9.9	10.6

Notes: Statistics are weighted means from the 2022 Survey of Consumer Finances; dollar values are in 2022 dollars. Meaningful liquidity is liquid assets of at least one quarter of income (LIQ $\geq$ INCOME/4, three months). Co-holders have positive total debt and meaningful liquidity. Debt holders without meaningful liquidity have positive total debt and LIQ $<$ INCOME/4. Liquid-asset holders without debt have DEBT ≤ 0 and LIQ $\geq$ INCOME/4. Education is the SCF category code (1–7 below high school completion; 8 high school; 9 some college; 10–11 associate degree; 12 bachelor's degree; 13 master's degree; 14 doctorate or professional degree); higher values indicate more education, and the means indicate direction only.

that co-holders take on debt to hold higher liquidity. The only households that outclass them are those without debt. Another interesting result is that co-holders have higher education on average, lending less evidence to the claim that households co-hold for a lack of financial literacy. This is not to say that financial literacy does not play a role, but rather that part of the story might also be explained by savvy decision making[4].

The data documents a number of important facts. Most importantly, co-holding is not just a credit card phenomenon, mortgage and installment debt co-holding may actually be the more costly side. Additionally, co-holding tends to be more predominant near or just before retirement and persists late in life. The companion paper is motivated by these two facts, studying mortgage co-holding quantitatively in a retirement life-cycle model. Since a great share of the literature studying co-holding has focused on credit cards, the next section of this paper builds a model to understand how well rational decision makers under standard assumptions can explain the phenomenon.

3 Baseline: Modeling Closed-ended Loans

The model tells a story that could fit a number of situations, two of which I highlight here. First, is the story of a homeowner who mortgages a home and anticipates a future sale. The home itself is an illiquid asset. If in the intermittent period between ownership and sale the owner anticipates needing access to liquidity, their mortgage is typically the cheapest form of debt. If the homeowner pays off their mortgage too quickly they may need to re-tap their home value. Whether through refinancing, reverse mortgaging, or home equity loans, these types of debt are typically more costly than a primary mortgage. Even refinancing is costly because it generally comes with fees. Therefore, the homeowner faces a tradeoff between paying down their mortgage to decrease interest paid, and the availability of liquidity now. This story is studied in detail in the context of retired age households in the companion paper.

Second is the story of a medical student with little financial resources. Assuming they have a good chance of graduating and finding employment, they anticipate very high future salaries. However, they are likely interested in consuming more now while they wait for that payoff, so they take out a student loan. Since a large share of the med student's endowment is only available to them in the future, even when they start earning income from residency, they are better off staying liquid now and paying off their loans once they receive higher salaries. Since student loans often charge preferential interest rates, the student could even

4. See Lusardi and Tufano (2009), Lucas (2015), Lusardi, Michaud, and Mitchell (2017), and Gathergood and Weber (2014) for the role of financial literacy.

benefit from holding the cash benefit of their loans in liquid savings because once again, re-tapping their future endowment will be costly.

In both these stories, what I'm discussing are cases where people secure closed-end loans while awaiting future endowments. If the future endowment is large enough and refinancing options are costly, then there is a liquidity option value to debt headroom that can outweigh the costs of interest payments.

3.1 Environment

The model has three time periods $t = \{1, 2, 3\}$. In the first period, households receive an endowment, ω_1, they save through a liquid asset, a_2, that earns a return, r, and they choose a loan benefit, d_2. In the second period, households do not receive any endowments, their resources are based solely on their first period portfolio, $\{a_2, d_2\}$. Households make a loan payment, m_2, and a savings decision, a_3, in the second period. In the third period, they receive another endowment, ω_3. Since it is the final period, there is no incentive to save (i.e., $a_4 = 0$), however, households must resolve any outstanding loan balances with the payment m_3.

Loan contracts in this model are closed-end installment loans with an amortizing repayment schedule, similar to a mortgage. For now I assume that it is impossible to refinance and there are no other forms of debt. I will relax these assumptions later. The repayment schedule can be calculated using a standard amortization

$$\underline{m}_t = d_t \frac{r_m(1 + r_m)^{4-t}}{(1 + r_m)^{4-t} - 1} \tag{4}$$

and household debt evolves according to

$$d_{t+1} = (1 + r_m)d_t - m_t \tag{5}$$

$$d_4 = 0. \tag{6}$$

To allow households some flexibility, $\underline{m}_t$ is only imposed as a minimum repayment. In the event that $m_t > \underline{m}_t$ is chosen, there is no penalty for overpayment[5]. Furthermore, I assume the amortization schedule is re-adjusted based on the balance at the start of each period, d_t[6]. This is a simplifying assumption that frees the minimum payment from being tracked as a

5. Some mortgage loans structure in early payment penalties (Beltratti, Benetton, and Gavazza 2017).

6. This is also an assumption because many closed-end loans have fixed payments that do not depend on balance. Typically, even if a mortgage is over-paid, it does not decrease the minimum payment though it still pays down the balance (McCollum, Lee, and Pace 2015).

state and instead structures it into the contract[7]. Regardless, the assumption ensures that if in each period the household makes minimum installments, $m_t = \underline{m}_2$, the loan will be paid back in full.

3.2 The Household Problem

Households earn utility from consumption in each period according to

$$U(c_1, c_2, c_3) = u(c_1) + \beta u(c_2) + \beta^2 u(c_3). \tag{7}$$

where u is assumed to be increasing, concave, differentiable, and $\lim_{c\to 0} u(c) = -\infty$. The budget constraints can then be written as

$$c_1 = \omega_1 - a_2 + d_2 \tag{8}$$
$$c_2 = (1+r)a_2 - m_2 - a_3 \tag{9}$$
$$c_3 = \omega_3 + (1+r)a_3 - m_3. \tag{10}$$

Note, since d_2 is the cash benefit from the loan, it shows up explicitly in the first period constraint. To reduce the number of constraints, equations 8-10 can be re-written in terms of current and future debt by substituting 5. Setting $m_t = (1+r_m)d_t - d_{t+1}$, the constrains are written as

$$c_1 = \omega_1 + d_2 - a_2 \tag{11}$$
$$c_2 = (1+r)a_2 - (1+r_m)d_2 - a_3 + d_3 \tag{12}$$
$$c_3 = (1+r)a_3 - (1+r_m)d_3 + \omega_3. \tag{13}$$

Since there is a minimum payment required on debt, removing m_t from the borrowing constraints requires a specification about debt constraints. In particular, because there is a lower bound on m_2, there will be an upper bound on d_3 of the form

$$d_{t+1} \leq (1+r_m)d_t - \underline{m}_t(d_t) \equiv \bar{d}(d_t). \tag{14}$$

Furthermore, since there is a borrowing margin, $a_t \geq 0$ is enforced.

Because the model is meant to explain why co-holding is optimal even when debt is costly, I will assume throughout that $r_m > r$. For this reason, there is a non-negative debt constraint

7. Given that this is a three-period model this actually is realistic. The companion paper studies a related model with 15 time periods. This assumption helps to reduce the computational burden while still maintaining a level of realism.

that can bind since savers would prefer to set $d_t < 0$, and earn $r_m > r$, as opposed to $a_t > 0$, earning r. For convenience, I summarize the full set of boundary constraints in 15[8]

$$a_2 \geq 0,\ a_3 \geq 0,\ d_2 \geq 0,\ d_3 \geq 0,\ m_2 \geq \underline{m}_2,\ m_3 \geq \underline{m}_3. \tag{15}$$

The households problem can then be specified as

$$\max_{c_1,c_2,c_3,a_2,a_3,d_2,d_3} U(c_1, c_2, c_3) \tag{16}$$

subject to equations 11-13, and the constraints in 15.

Since this is just a three period model, the intertemporal decisions of the model can be studied explicitly by solving a Lagrangian. Substituting the budget constraints 11-13 into the utility function, I can write the following Lagrangian

$$\begin{aligned} L(\boldsymbol{x}) = {} & u\big(\omega_1 + d_2 - a_2\big) \\ & + \beta u\big((1+r)a_2 + d_3 - a_3 - (1+r_m)d_2\big) + \beta^2 u\big((1+r)a_3 - (1+r_m)d_3 + \omega_3\big) \\ & + \lambda_2 a_2 + \beta\lambda_3 a_3 + \theta_2 d_2 + \beta\theta_3 d_3 - \beta\phi\big(d_3 - \bar{d}(d_2)\big) \end{aligned} \tag{17}$$

where $\boldsymbol{x} = (a_2, a_3, d_2, d_3, \lambda_2, \lambda_3, \phi, \theta_2, \theta_3)$, λ_2, λ_3, are the Lagrange multipliers for credit constraints, θ_2, θ_3, are the Lagrange multipliers for non-negative debt constraints, and ϕ is the Lagrange multipliers for debt headroom. I assume that $\beta(1+r) = 1$ throughout for convenience. The Euler equations that dictate behaviour in this model can then be written as

$$u'(c_1) = u'(c_2) + \lambda_2 \tag{18}$$

$$u'(c_2) = u'(c_3) + \lambda_3 \tag{19}$$

$$\theta_2 + \beta\phi\bar{d}'(d_2) + u'(c_1) = \beta(1+r_m)u'(c_2) \tag{20}$$

$$\theta_3 + u'(c_2) = \beta(1+r_m)u'(c_3) + \phi. \tag{21}$$

with the associated Karush-Kuhn Tucker (KKT) conditions

$$\lambda_t a_t = 0,\ \lambda_t \geq 0 \tag{22}$$

$$\theta_t d_t = 0,\ \theta_t \geq 0 \tag{23}$$

$$\phi\left(d_3 - \bar{d}(d_2)\right) = 0,\ \phi \geq 0 \tag{24}$$

$$\theta_t \phi = 0. \tag{25}$$

8. $c_t > 0$ is also a boundary constraint but it is redundant here because of the standard assumptions on utility.

Equations 18 and 19 are the same that appear in standard consumption-savings models with incomplete markets and they regulate savings decisions in the usual way: if the non-borrowing constraint on liquid assets binds, then $a_{t+1} = 0$ and $c_t < c_{t+1}$. By contrast, equations 20 and 21 are non-standard and will regulate debt decisions here.

3.3 Interpreting Euler Equations

With $\beta(1 + r) = 1$, households optimally smooth consumption unless doing so would require that $a_t < 0$. However, since $\beta(1 + r_m) > \beta(1 + r) = 1$, if smooth consumption is optimal then for equation 21 to hold, θ_t must be positive. This of course makes sense because savers would prefer to earn $r_m > r$ on savings. However, since the term $\beta\phi\bar{d}'(d_2)$ appears on the left-hand side of 20, it is possible for equation 20 to hold without the desire to set $d_2 < 0$.

Debt headroom constraints will bind whenever $\phi > 0$, which can happen when households anticipate high future payouts but are starved for liquidity now. Note the intertemporal effects of the debt headroom constraint: since the loan contract in this model is closed-ended, the debt payment choice in period 1 affects the maximum debt a household can carry into period 3. The easiest way to see this is to assume that the household doesn't take out a loan in the first period. In this case, the household's maximum future debt choice in period 2 is zero. This allows for interpretations of 20 and 21. The left-hand sides of these equations can be thought of as the marginal values of debt and the right hand sides as the marginal costs of debt. For equation 21 this leads to a simple conclusion. The value of holding onto an additional dollar of debt is equal to the marginal value of additional consumption. However, this debt needs to be paid back with interest in the following period so the marginal cost is the value of forgone future consumption. Equation 20 differs because it takes into account the future value of debt headroom $\beta\phi\bar{d}'(d_2)$. Therefore, in the first period, the marginal value of a loan valued at d_2 is the additional consumption valued at $u'(c_2)$, plus the future value of debt headroom. The marginal cost again the value of forgone future consumption.

3.4 Co-Holding Solution

Next, I focus on the family of co-holding solutions and the conditions that guarantee them. These conditions will boil down to the relative magnitudes of ω_1 compared to ω_3. When the ratio of ω_3 to ω_1 is sufficiently large, the only way for the household to try to smooth consumption between periods 2 and 3 is to maintain high liquidity in a_2, which can be done by choosing a sufficiently high loan benefit, d_2.

I begin by defining co-holding explicitly as any solution where $a_2 > 0$ and $d_2 > 0$. By Karush-Kuhn-Tucker (KKT) conditions, $a_2 > 0$ and $d_2 > 0$ if and only if $\lambda_2 = \theta_2 = 0$.

Therefore, from 18 I can conclude that whenever co-holding exists, $c_1 = c_2$[9]. From 20, since $r_m > r$ and $\phi \geq 0$, it can also be concluded that $\phi > 0$, otherwise the system will be inconsistent. From the KKT conditions this implies necessarily that $d_3 = \bar{d}(d_2)$ and $\theta_3 = 0$. Then, for equations 19 and 21 to remain consistent, it must also be the case that $\lambda_3 > 0$ and $a_3 = 0$. Finally, since u is increasing and concave, 19 implies that $c_3 > c_2$. This provides a candidate solution so long as we can verify that 20 is satisfied with $\phi \geq 0$. The conditions that guarantee a coholding solution are provided in proposition 1

Proposition 1. *[Co-holding Solutions] The set of co-holding solutions is*

$$\mathbb{C} = \left[(c_1, c_2, c_3) \middle| c_1 = c_2 \ \& \ \frac{u'(c_2)}{u'(c_3)} = \frac{\beta^2(1+r_m)^2}{\beta(1+r_m) - (\beta(1+r_m)-1)(2+r_m)} \right].$$

Proof. See appendix D. □

The household's optimal choice of (c_1, c_2, c_3) will depend on the set of endowments $\{\omega_1, \omega_3\}$. Since co-holding is optimal whenever c_3 is sufficiently large compared to c_2, co-holding will depend on mostly on the relative difference between ω_3 and ω_1. In fact, to demonstrate when co-holding solutions exist from initial conditions, it is easy to conceive of cases where ω_3 is very large and ω_1 is quite small. In these cases, even if the household saves all of a_1 (doesn't consume in period 1), without holding debt, they would not be able to reach a point where c_2 is even close to c_3. Therefore, the only way to smooth out consumption for these types of households is to co-hold. Lemma 1 proves the initial conditions required to obtain a solution where $c_2 < c_3$.

Lemma 1. *$c_2 < c_3$ becomes optimal when*

$$\frac{\omega_3}{\omega_1} > \frac{1+r}{2+r}.$$

Otherwise, $c_2 = c_3$ is optimal.

Proof. This is fairly straight forward to prove. The moment the constraint $a_3 \geq 0$ binds, $a_3 = 0$ is optimal implying $c_2 = c_3$ and $d_2 = 0$, so we can write

$$c_1 = \omega_1 - \frac{1}{2+r}\omega_1 = \omega_3 = c_3$$

which simplifies to

$$\frac{\omega_3}{\omega_1} = \frac{1+r}{2+r}$$

9. It is easy to see that this will also be true for any solution. Since there is no second period endowment, $c_2 \leq 0$ unless $a_2 > 0$

If the ratio ω_3/ω_1 increases any further it is not possible to smooth consumption because it would require $a_3 < 0$. □

Since $r_m > r$, lemma 1 doesn't necessarily guarantee coholding. The wedge between c_2 and c_3 still needs to be large enough to satisfy the conditions in proposition 1, so this is not a sufficient co-holding condition. However, if the conditions in 1 are met then the household will set $a_3 = 0$. Since u is both increasing and concave, this also reveals that as ω_3/ω_1 increases, without holding any debt, the ratio of marginal utilities continues to grow. If it grows to the point highlighted in proposition 1, then the household will be a co-holder.

Proposition 2. *[Initial Conditions that guarantee co-holding] A household will choose to co-hold if and only if*

$$\frac{u'\left(\frac{1+r}{2+r}\omega_1\right)}{u'(\omega_3)} > \frac{\beta^2(1+r_m)^2}{\beta(1+r_m) - (\beta(1+r_m)-1)(2+r_m)}$$

Proof. See appendix D. □

Looking more closely at proposition 2, this says that once ω_3/ω_1 is sufficiently large such that households cannot sufficiently smooth consumption through liquid saving alone, borrowing becomes optimal. This is similar to the group of households that Kaplan, Violante, and Weidner (2014) identifies as the wealthy hand-to-mouth (WHtM): households with strong future resources but weak current cash flow. Thus, the connection between the two frameworks is that the households most likely to co-hold should be the WHtM[10]. The mapping between the two frameworks is close in terms of which households are involved, but portfolio choices are different. In a model without closed-end loans, these households are stuck at the borrowing constraint with $c_2 < c_3$ and present with high illiquid wealth and near-zero liquid assets. In the present model, the same households use the loan to relax the constraint and instead present as co-holders: positive liquid assets and positive debt. Co-holding is therefore not a separate puzzle from WHtM but a similar phenomenon observed through a richer set of financial instruments.

3.5 Numerical Illustration

I now present a simple numerical example that illustrates how optimal portfolio choices vary with households' endowments. For simplicity, I assume log preferences,

$$u(c) = \ln c, \tag{26}$$

10. Druedahl and Jørgensen (2018) come to a similar conclusion, suggesting that co-holding should be an important tool for HtM households.

and maintain the normalization $\beta(1+r)=1$ used above, so that $\beta = 1/(1+r)$. Throughout this section I fix the interest rates on liquid savings and on debt as summarized in Table 6.

Table 6: Parameter choices for numerical examples

	Symbol	Value
Risk-free savings rate	r	0.05
Loan (debt) interest rate	r_m	0.10
Discount factor	β	$1/(1+r)$

The model is solved as a system of non-linear equations. Figure 4 shows the regions in which co-holding is optimal. Notably, to generate co-holding, the household need not have

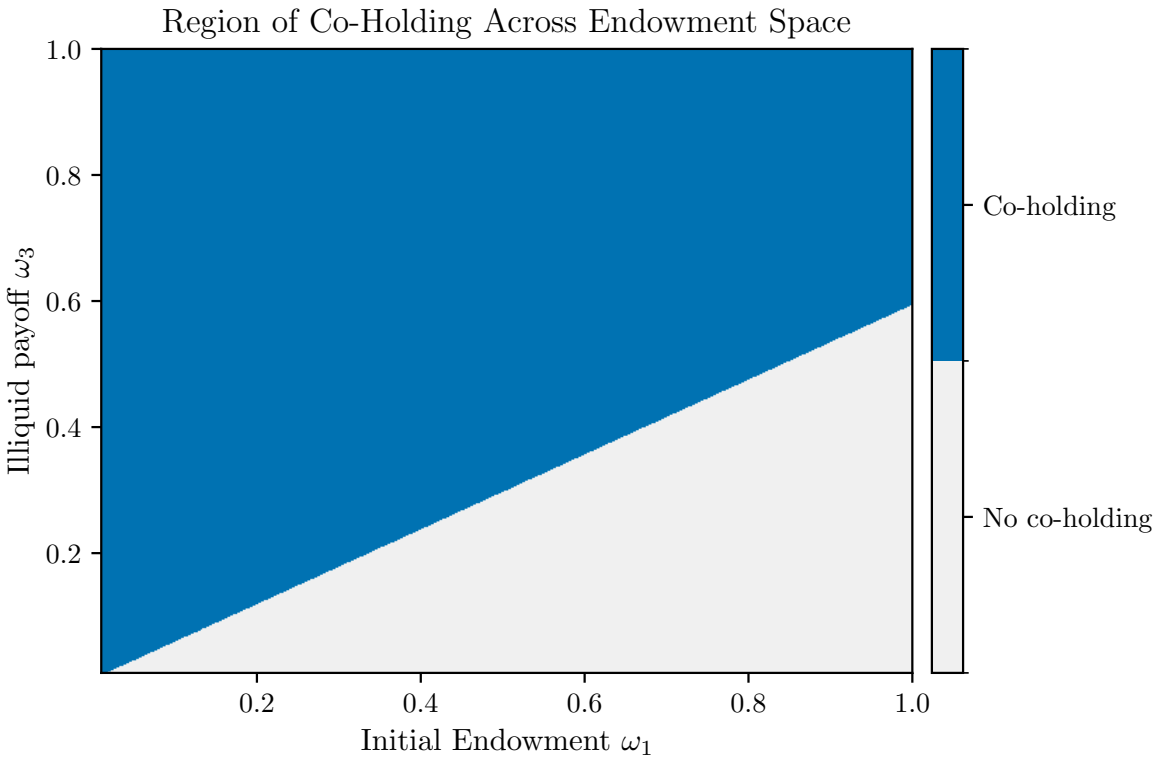


Figure 4: Co-holding Region in the Baseline

Note: This plot shows the regions of co-holders ($a_2 > 0$, $d_2 > 0$) vs. asset holders ($a_2 > 0$, $d_2 = 0$) simulated numerically from the simple model parametrized by table 6 plotted across the endowment space. The region in blue indicates co-holders. In simulation, the endowment space varies from 0 to 1 for both first period (horizontal axis) and third period (vertical axis) endowments.

endowments where $\omega_3 > \omega_1$. This makes sense because the household must rely on their endowment of ω_1 to last across the first two periods by saving. Furthermore, in the previous section, for co-holding to occur, it had to be optimal for the household to choose $c_1 = c_2 < c_3$.

Figure 5 supports this fact as the dark shaded region is the region in which $c_1 = c_2 = c_3$. Then, along the same line across the endowment space that separates the savers from the

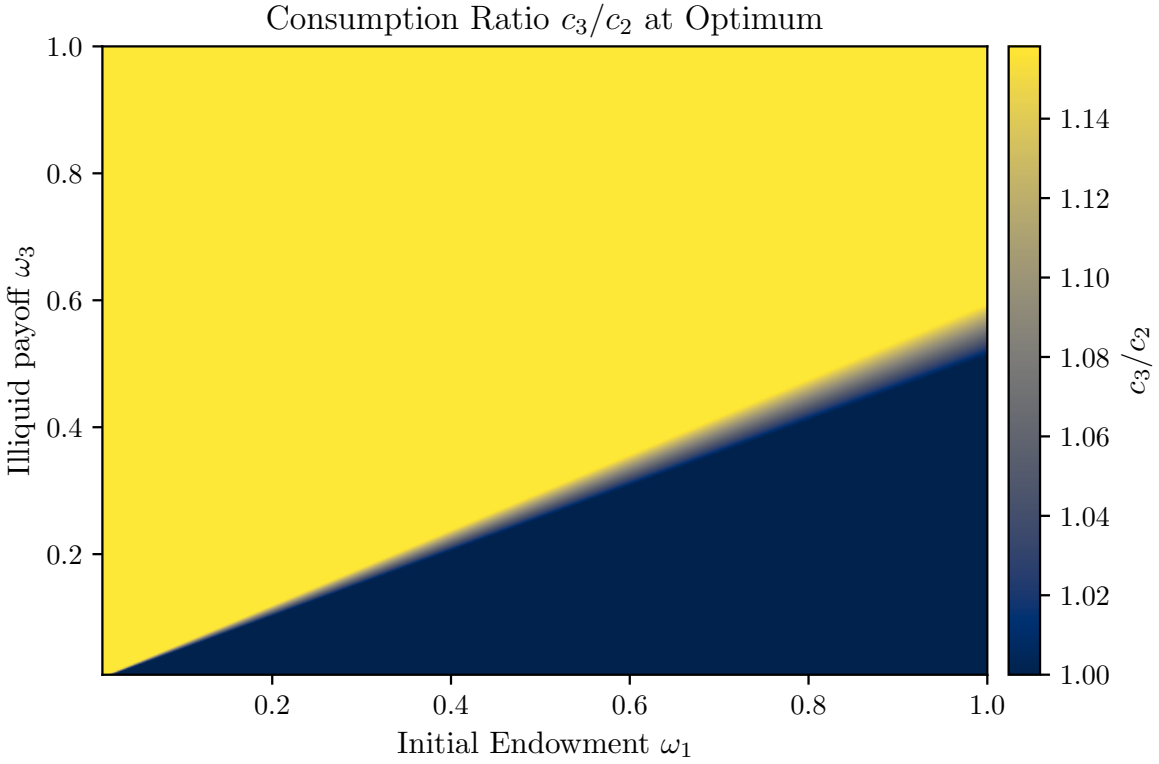


Figure 5: Future to Current Consumption Ratios

Note: This plot shows a heatmap of the ratio of third period to first period consumption from the simple model parametrized by table 6 plotted across the endowment space. Warmer regions are denoted by brighter colours. In simulation, the endowment space varies from 0 to 1 for both first period (horizontal axis) and third period (vertical axis) endowments.

co-holders does it also separate consumption smoothers. However, there does exist a margin over which $c_3 > c_2$, and co-holding is not optimal depicted by the gradient between the dark and light shaded regions. This occurs because of the spread between the two interest rates. Some households can be constrained by $a_3 \geq 0$, but not heavily enough that borrowing at the rate, r_m, is worth it. Then, as ω_3 increases relative to ω_1, so too does the ratio c_3/c_2.

Finally, to show how portfolio choices evolve relative to endowments, figure 6 shows how the chosen ratio of debt-to-assets changes across the endowment space. Here, debt is the loan value chosen by the household and assets are the total value of household assets in the first period, including the value of the future endowment, ω_3. Consistent with the rest of the story, households choose higher leverage as the illiquid endowment grows relative to the liquid endowment. When the ratio of ω_3 to ω_1 is very large, households choose leverages in excess of 60% of total assets.

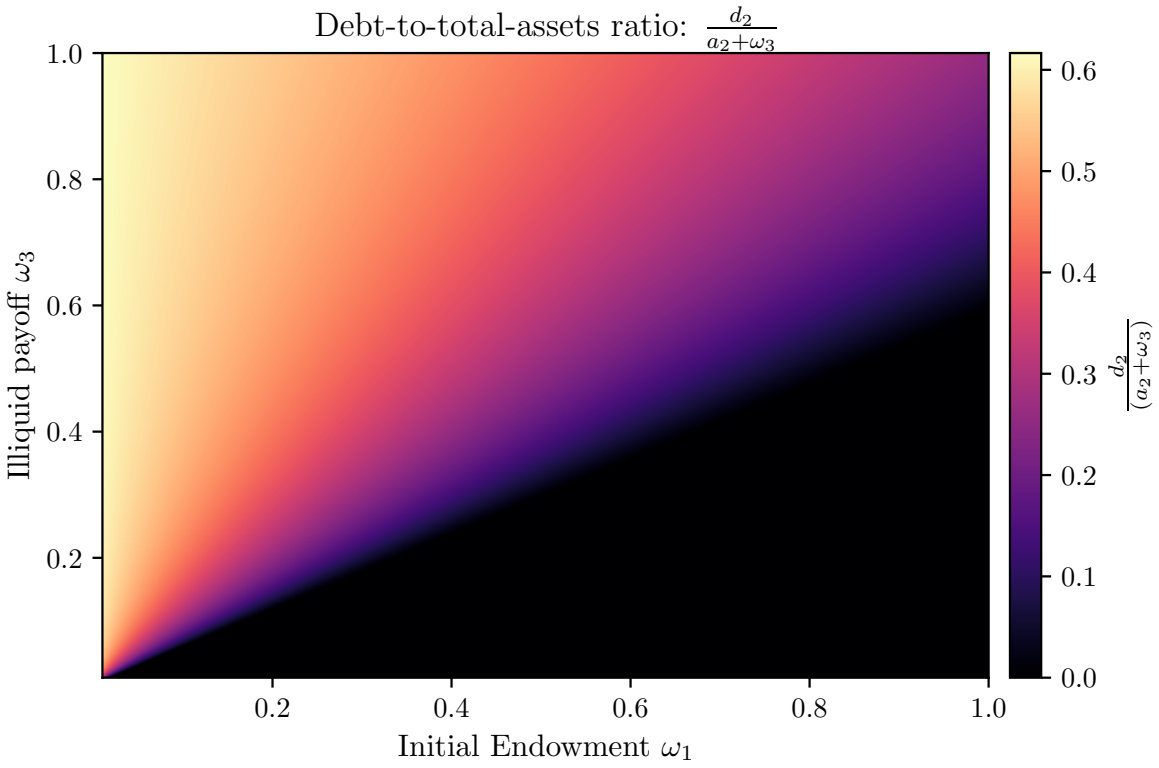


Figure 6: Optimal Leverage Choices

Note: This panel shows a heatmap of the optimal debt to total asset ratio from the simple model parametrized by table 6 plotted across the endowment space. Total assets are define as liquid assets chosen in the first period plus third period endowment. Warmer regions are denoted by brighter colours. In simulation, the endowment space varies from 0 to 1 for both first period (horizontal axis) and third period (vertical axis) endowments.

3.6 Comparison to a Net-Worth Model

Now, I re-frame the model into a more traditional setting with a single portfolio choice, a_2. To keep this consistent with the baseline, I impose a piecewise interest rate on savings ρ such that

$$\rho = \begin{cases} r & \text{if } \ a \geq 0 \\ r_m & \text{if } \ a < 0 \end{cases}. \tag{27}$$

It would also be desirable to impose a fixed amortized repayment schedule on loans, but since there is no second period endowment, second period consumption would turn negative for first period borrowers[11]. Without the schedule, households entering the third period with debt need to pay off their debts anyways so fixed payments aren't important qualitatively. Otherwise, apart from the absence of d_t, the model remains the same.

The household problem can then be re-formulated as

$$\max_{c_1,c_2,c_3,a_2,a_3} u(c_1) + \beta u(c_2) + \beta^2 u(c_3) \tag{28}$$

subject to

$$c_1 = \omega_1 - a_2 \tag{29}$$

$$c_2 = (1+\rho)a_2 - a_3 \tag{30}$$

$$c_3 = \omega_3 + (1+\rho)a_3. \tag{31}$$

Written as a Lagrangian,

$$L(a_2, a_3, \lambda) = u\big(\omega_1 - a_2\big) + \beta u\big((1+\rho)a_2 - a_3\big) + \\ \beta^2 u\big(\omega_3 + (1+\rho)a_3\big) + \beta\lambda\big(a_3 - \bar{a}(a_2)\big) \tag{32}$$

Keep in mind here that due to the interest rate schedule, the problem is not differentiable at $a_t = 0$. For this reason, I specify a separate problem for savers and for borrowers[12]. With

11. A fixed repayment schedule in this case would necessitate that $(1+r_m)a_2 < a_3$ in the second period, and so $c_2 = (1+r_m)a_2 - a_3 < 0$.

12. The break in interest rates means there is a potential corner solution at $a_2 = 0$. If initial endowments are decreased down to the point where $a_2 = 0$, decreasing them further for some margin, the household will hold $a_2 = 0$ before they start borrowing.

$\beta(1+r)=1$, for savers the optimality conditions are

$$u'(c_1) = u'(c_2) \tag{33}$$
$$u'(c_2) = u'(c_3) \tag{34}$$

and for borrowers

$$u'(c_1) = \beta(1+r_m)u'(c_2) \tag{35}$$
$$u'(c_2) = \beta(1+r_m)u'(c_3). \tag{36}$$

This problem will generate three types of households, saver-savers, borrower-borrowers, and saver-borrowers. Saver-borrowers are households who are first-period savers and second-period borrowers. Likewise, borrower-borrowers and saver-savers are borrowers and savers respectively across all periods. There are no borrower-savers in this model because there is no second period endowment[13]. Saver-borrowers are perhaps this model's closest version of co-holders. However, borrower-borrowers in the net-worth model would also be co-holders in the baseline[14]. In other words, anyone with debt in the baseline also saves. This reveals perhaps another important category of co-holder. Some co-holders optimally borrow at margins higher than their first period endowment — borrower-borrowers here. These co-holders are more likely to be constrained by incomplete markets because they may not have enough initial assets to qualify for loans. This will become more relevant when I extend the model to home purchases in the companion paper.

In a deterministic setting with complete markets, a net-worth or single-asset model will explain mostly the same behaviour as the baseline here. Qualitatively, the results are the same. Saver-borrowers, borrower-borrowers, and co-holders choose third period consumption such $c_1 = c_2$ and $c_2 < c_3$. In the single asset model this is true because $\beta(1+r_m) > 1$. Quantitatively, since co-holders have to take on debt in the first period and then pay interest, there can be some differences. Figure 7 shows quantitatively how a net-worth model compares to the baseline example previously. When households have significantly high third period endowments but low first period endowments, households in the baseline model become co-holders. Households in the net-worth model become borrowers in both first and second periods in this range because they do not have enough endowment to split in the first two periods.

13. First period borrowers and second period savers would necessarily have negative second period consumption since $a_3 > 0$ and $a_2 < 0$ implies $c_2 = (1+\rho)a_2 - a_3 < 0$.

14. Since there is no second period endowment, households in the baseline that borrow need to carry at least some savings into the second period if they want to consume. In reality there are people who only have debt and no savings. But households here are not able to default on debt so this is more akin to a model about secured debt.

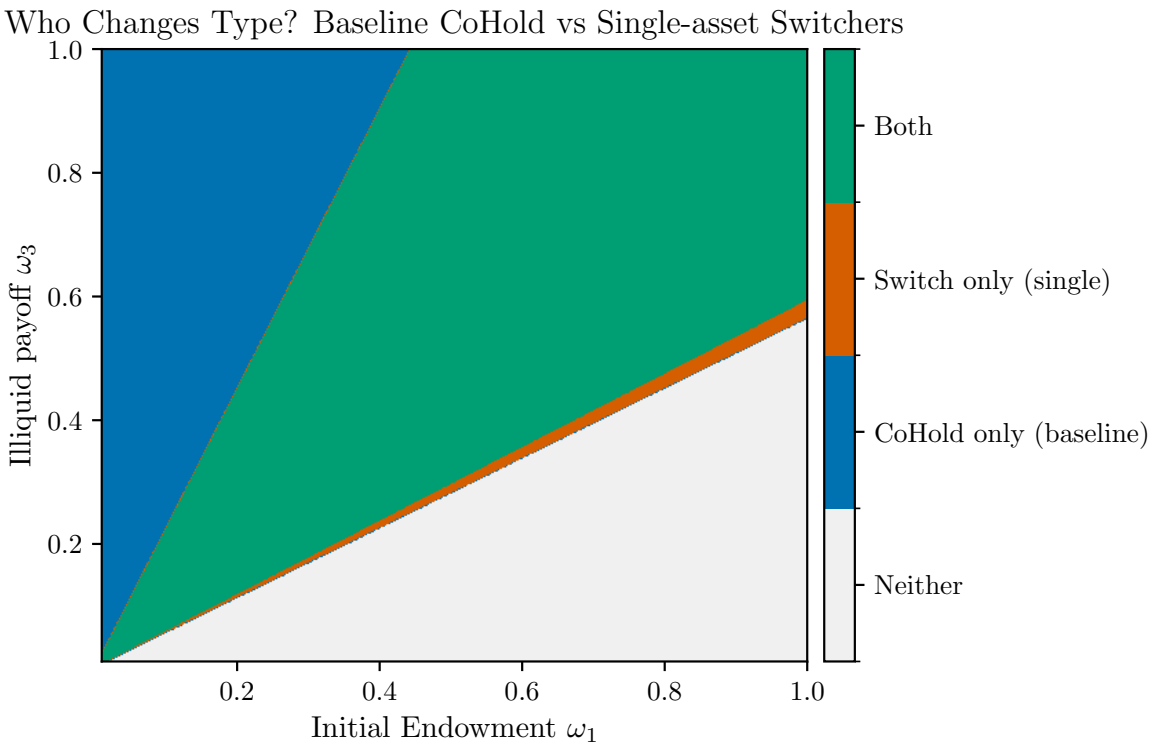


Figure 7: Type Comparison Between both Model

Note: This panel shows the simple co-holding model compared to a consolidated net-worth model parametrized by table 6 plotted across the endowment space. The shaded orange region (switch only) coincides with the region from 5 where $c_3 > c_2$ but there is no co-holding ($a_2 > 0$, $d_2 = 0$). In other words, these households are constrained in the co-holding model but do not choose to co-hold whereas in the consolidated model, they save in the first period ($a_2 > 0$) borrow in the second period ($a_3 < 0$). The green region (both) coincides with the region where the household is a saver-borrower ($a_2 > 0$, $a_3 < 0$) in the net-worth model, but a co-holder ($a_2 > 0$, $d_2 > 0$) in the baseline. The blue region (co-holders) then coincides with households who are borrower-borrowers ($a_2 < 0$, $a_3 < 0$) in the net-worth model and co-holders in the baseline.

While this quantitative difference doesn't seem important for now, portfolio composition does start to become relevant after a loan has been chosen. When I add uncertainty later, households with the same net worth but different portfolios will behave differently.

3.7 The Importance of Portfolio Composition

Before introducing any uncertainty into the model, the baseline model can already be used starting from the second period with different states to see why portfolio composition matters. Suppose there are two households, $\{A, B\}$, who arrive at $t = 2$, with $a_2 = a_I$, and $d_2 = d_I$, where $I = \{A, B\}$ respectively, such that $(1 + r)a_A - (1 + r_m)d_A = (1 + r)a_B - (1 + r_m)d_B$. To focus on the portfolio, I assume that future endowments are the same for both. In this case both these households would have the same net worth, however, it is not hard to see that the household with higher debt is actually a weakly preferred state. For instance, suppose $d_A > d_B = 0$ and $\omega_3 > (1 + r)a_A - (1 + r_m)d_A = (1 + r)a_B$. First-order conditions would optimally require that $u'(c_2) = u'(c_3) + \lambda_3$. Since $d_B = 0$, the best household B can do is set $d_3 = 0$. By contrast, household A has flexibility because they can set $d_3 \in [0, \bar{d}(d_A)]$. Therefore, portfolio composition matters because it affects exposure to constraints: leveraged households with the same net-worth have more liquidity to manage short-term expenses.

While this simple example does demonstrate the intuition, there are two practical problems. First, two households with the same net worth in the first period would always make the same decisions. Therefore, in a deterministic model, it is not actually possible for two households to arrive in the second period with different portfolios and the same net-worth. In a deterministic setting, net-worth is a sufficient statistic. This changes once households face uncertain outcomes. Suppose I extend the model to allow for some random expense shock, e, to arrive in period 2. Then, the new budget constraint becomes

$$c_2 = (1 + r)a_2 - (1 + r_m)d_2 - a_3 + d_3 - e. \tag{37}$$

Households form expectations over e, and households with the same ex-post net worth (after expense shocks are drawn) can have different portfolios. Quantitatively, these households will behave differently depending on their demand for liquidity, a feature which net-worth models cannot capture.

The second problem is that heterogeneity in behaviour results directly from feasibility. Households that make different decisions do so because they are constrained from making the otherwise optimal choice. This problem is simply a result of the 3-period model. With uncertainty and more intermediate periods, households still make different decisions based on their portfolios not because they are forced by feasibility, but to provide themselves with

insurance to avoid future states becoming infeasible. This outcome is more clear in the companion paper, where I explore a more realistic model in this family with 15 periods.

In reality, households have a choice over many different portfolio objects, and closed-end loans are far from the only option. Even in the case where households do have closed-end loans, these are typically not strictly closed, it's always possible for a lender to buy out the loan and offer new terms. However, since closed-end loans are typically less risky for lenders, they tend to offer the cheapest interest rates. The tradeoff for households is a loss of flexibility. For these reasons, it is important to see how closed-ended loan frictions hold up when households are given access to these options. The next section of the paper explores co-holding frictions in the presence of different objects.

4 Extending the Baseline Model

Now that it is understood how the model works in a basic deterministic setting, this section will extend the model to show that the main implications of the simple mechanism still hold even with more realistic environments. This will also serve as a means of justifying some of the assumptions in the companion paper. Every one of these extensions play a role, however, some simplifying assumptions lead to only small quantitative differences.

First, I introduce uncertainty to the model formally and show how the model can generate heterogeneity in portfolios for households with the same net-worth. This is the main advantage this type of model buys that net-worth models cannot. Second, the reality is that households usually have different types of savings and debt. In particular, I will introduce two types of portfolio objects that offer households flexibility to show how the model behaves here; open-end credit and low-liquidity high-return assets. In comparison to closed-end loans, open-end credit, such as a credit card, offers households a borrowing limit that can be accessed at any time. Third, households that have loans tend to be given opportunities to refinance at a cost, bu this is left to study in detail in the companion paper.

4.1 Adding Expense Shocks

Suppose now that the household faces some future expense risk of the form $e \sim \mathcal{F}$. The household problem changes because now expected utility is maximized conditional on the expense shock. However, since there are possibly infinite histories, the Lagrangian becomes a lot more notationally intensive, so I re-formulate this as a dynamic program.

The household state is time dependent, so I summarize it with the vector $\boldsymbol{x}_t$. In the first period, the household has a single state variable, $\boldsymbol{x}_1 = \omega_1$. In the second period, the

household has three state variables, $\boldsymbol{x}_2 = (a_2, d_2, e)$. In the final period, there are again three state variables $\boldsymbol{x}_3 = (a_3, d_3, \omega_3)$. In general, the individual problem can be written as the following Lagrangian

$$V(\boldsymbol{x}_t) \equiv \max_{c_t, a_{t+1}, d_{t+1}} \Big\{ u(c_t) + \beta E[V(\boldsymbol{x}_{t+1})] + \\ \mu_t[y_t(\boldsymbol{x_t}) - c_t - a_{t+1} + d_{t+1}] + \lambda_t a_{t+1} + \theta_t d_{t+1} + \phi_t \left(\bar{d}(d_t) - d_{t+1} \right) \Big\} \quad (38)$$

where $y_t(\boldsymbol{x_t})$ is the household net worth at the start of the period

$$y(\boldsymbol{x_t}) = \begin{cases} \omega_1 & \text{if } \ t = 1 \\ (1+r)a_2 - (1+r_m)d_2 - e & \text{if } \ t = 2 \\ (1+r)a_3 - (1+r_m)d_3 + \omega_3 & \text{if } \ t = 3. \end{cases} \quad (39)$$

The first-order conditions are the same, except in the first period, households take expectations over second period utility. To help understand the intuition, I derive the Euler equations to highlight the second period envelope theorem condition with respect to debt. The first-order conditions are

$$u'(c_t) = \beta E[V_a(\boldsymbol{x}_{t+1})] + \lambda_t \quad (40)$$

$$\theta_t + u'(c_t) = -\beta E[V_d(\boldsymbol{x}_{t+1})] + \phi \quad (41)$$

where I've made the substitution $\mu_t = u'(c_t)$. The four relevant envelope theorem conditions are then

$$V_a(\boldsymbol{x}_2) = \mu_2(1+r) \quad (42)$$

$$V_a(\boldsymbol{x}_3) = \mu_3(1+r) \quad (43)$$

$$V_d(\boldsymbol{x}_2) = -\mu_2(1+r_m) + \phi(\bar{d}'(d_2)) \quad (44)$$

$$V_d(\boldsymbol{x}_3) = -\mu_3(1+r_m). \quad (45)$$

The derivative of the value function with respect to debt (equations 44) also includes the third period debt constraint. This is another way to think about the value of debt headroom. The marginal value to paying off more debt today earns higher utility from consumption but

harms liquidity. Together these translate to the Euler equations

$$u'(c_1) = E[u'(c_2)] + \lambda_2 \tag{46}$$

$$u'(c_2) = u'(c_3) + \lambda_3 \tag{47}$$

$$\theta_2 + \beta E[\phi]\bar{d}'(d_2) + u'(c_1) = \beta(1 + r_m)E[u'(c_2)] \tag{48}$$

$$\theta_3 + u'(c_2) = \beta(1 + r_m)u'(c_3) + \phi. \tag{49}$$

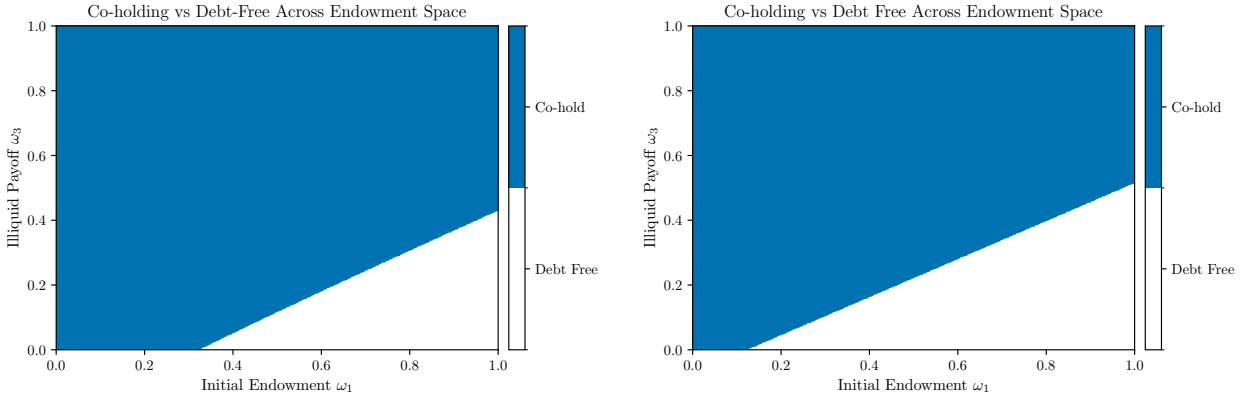
Now that the second period value of resources net of expenses includes an expense shock, both the expectation of future utility and the expected future value of third period debt headroom have to be taken into account. Co-holding incidence is also not as easy to establish now because of the stochastic second period. Proposition 3 summarizes the necessary conditions that dictate co-holding.

Proposition 3. *Let $\bar{e}$ be the upper bound of the support for $\mathcal{F}$. For a given choice of a_2, $d_2 > 0$ is optimal only if for some $e^* < \bar{e}$, the household is debt constrained in the second period.*

Proof. Proof in appendix D □

While Proposition 3 only establishes necessary conditions for co-holding, establishing existence follows from proposition 1. Consider a deterministic model where $e = \underline{e}$. If the household optimally co-holds here, then with e uncertain, since $e \geq \underline{e}$, it will always be the case that $u'(c_2(\underline{e})) \leq E[u'(c_2)]$, thus this household would also co-hold in the stochastic model.

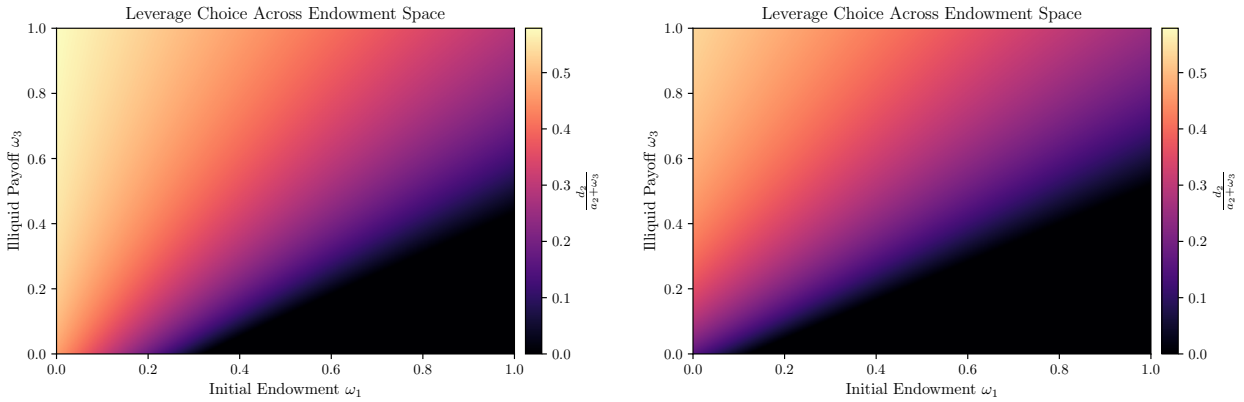
Compared to the deterministic case, proposition 3 is not quite as satisfying because it doesn't provide a clean connection between endowments and co-holding. Due to stochastic expense shocks, co-holding incidence will depend on the both the distribution of expense shocks and functional forms. Therefore, any further characterization of the co-holding conditions will be too general to add any value. Proposition 3 does however allow us to study how expense shock risk will affect decisions. To understand the intuition, suppose that the household has chosen a level of a_2 whether optimal or not. For that given choice, the higher $\bar{e}$ is, the more likely the household is to brush up against the debt headroom constraint, and thus the more likely the household is to choose $d_2 > 0$. For households who are constrained when drawing $\bar{e}$, the higher the likelihood of drawing $e > e^*$, the higher debt level they will choose. However, this can only be true if higher debt is still feasible within the household's budget constraint. Therefore, there is some doubt as to whether expense risk should increase or decrease leverage.



(a) Stochastic Model (b) Deterministic Model

Figure 8: Co-holding Comparison Between Baseline and Stochastic Models

Note: These panels show the regions of co-holders ($a_2 > 0$, $d_2 > 0$) vs. asset holders ($a_2 > 0$, $d_2 = 0$) from the stochastic model in panel (a) compared to the deterministic model in panel (b). Both models are parametrized by table 6, but the stochastic model takes uniform expense shocks, $e \in (0, 0.4)$ and the deterministic model takes the average, $\bar{e} = 0.2$. The region in blue indicates co-holders. In simulation, the endowment space varies from 0.3 to 2 for both first period (horizontal axis) and third period (vertical axis) endowments. The endowments vary across the interval $(0.3, 2)$ and are normalized using the transformation $\hat{\omega} = (\omega - \underline{\omega})/(\bar{\omega} - \underline{\omega})$ so the axes read $(0, 1)$ for consistency.



(a) Stochastic Model (b) Deterministic Model

Figure 9: Debt-to-Asset Choices

Note: These panels show the heatmaps of debt-to-asset ratios from the stochastic model in panel (a) compared to the deterministic model in panel (b). Both models are parametrized by table 6, but the stochastic model takes uniform expense shocks, $e \in (0, 0.4)$ and the deterministic model takes the average, $\bar{e} = 0.2$. Brighter regions indicate warmer regions in the plots. In simulation, the endowment space varies from 0.3 to 2 for both first period (horizontal axis) and third period (vertical axis) endowments. The endowments vary across the interval $(0.3, 2)$ and are normalized using the transformation $\hat{\omega} = (\omega - \underline{\omega})/(\bar{\omega} - \underline{\omega})$ so the axes read $(0, 1)$ for consistency. Assets are computed as the first period saving choice combined with third period endowment.

Figures 8 and 9 demonstrate an example of how an increase in leverage could increase leverage and co-holding under a mean-preserving spread. The panels on the left (figures 8a and 9a) reflect the model under a uniform distribution with $e \in [0, 0.4]$, and the panels on the right reflect a deterministic model with $e = 0.2$. Both co-holding incidence and leverage choices are lower in the deterministic model because risk increases the demand for liquidity.

For the rest of the paper, I retain expense shocks; despite the name, they are not computationally or analytically expensive to include and they do provide an additional layer worth studying. Next, the paper will study how co-holding behaviours change when households have more portfolio options.

4.2 Explicit Lines of Credit

Households with debt are often observed with different types of debt. Lusardi, Mitchell, and Oggero (2020) finds that households with a mortgage are much more likely to have other types of secured debt (e.g., HELOC, car loan). The most typical form of consumer debt is credit card debt, though this is often also the most expensive debt. But credit card loans in are different from the loan studied in the baseline because they are open-ended. There is a strand of literature that works on understanding why households revolve credit card debt with low interest savings and this has been referred to as the credit debt puzzle (Fulford 2015; Telyukova 2013; Zinman 2007; Bertaut, Haliassos, and Reiter 2009). Households will rationally choose to hold open loans or credit when these objects have different liquidity options. These types of frictions could be imposed to generate situations where households co-hold simultaneously low interest debt, high interest loans, and higher interest credit however, the purpose here is to show that even when when households have flexibility to hold open loans, the results still hold if open loans or credit is more expensive than closed-end loans. This extension then coincides more with mortgage debt home equity loans. As households pay down their mortgages, they usually gain access to home-equity lines of credit, and occasionally reverse mortgages. While these loans are typically open and offer flexibility, they tend to feature higher interest rates that makes closed-end mortgages cheaper.

Since credit and liquid assets have the same liquidity value and I assume no frictions in credit markets, households will never elect to co-hold savings and credit simultaneously. Instead of having an explicit variable for credit, I relax the constraint on liquid assets and allow households to borrow up to their natural borrowing limits. If the credit rate were lower or equal to the rate on closed-end loans, there would be no incentive to take out closed-end loans. This assumption also respects the realism of how closed-end loans are different from open loans. Closed-end loans are more strict in structure and are less risky for lenders. Thus,

to offer credit, lenders require a risk premium in the form of higher rates. I denote the interest rate on credit by the term r_b and set r_b such that $r < r_m < r_b$.

Rewriting the Bellman equation to reflect these changes, keeping in mind it will no longer be differentiable around the point $a_{t+1} = 0$, the Lagrangian can be written as

$$V(\boldsymbol{x}_t) \equiv \max_{c_t, a_{t+1}, d_{t+1}} \Big\{ u(c_t) + \beta E[V(\boldsymbol{x}_{t+1})] + \\ \mu_t[y_t(\boldsymbol{x_t}) - c_t - a_{t+1} + d_{t+1}] + \theta_t d_{t+1} + \phi_t \left(\bar{d}(d_t) - d_{t+1} \right) \Big\} \quad (50)$$

where I drop the term λ_t because the natural borrowing limit never binds[15]. The interest rate conditional on savings is written as

$$R_t = \begin{cases} r & \text{if } \ a_t \geq 0 \\ r_b & \text{if } \ a_t < 0. \end{cases} \quad (51)$$

The Euler equations now become

$$u'(c_1) = \beta(1 + R_2)E[u'(c_2)] \quad (52)$$

$$u'(c_2) = \beta(1 + R_3)u'(c_3) \quad (53)$$

$$\theta_2 + \beta E[\phi]\bar{d}'(d_2) + u'(c_1) = \beta(1 + r_m)E[u'(c_2)] \quad (54)$$

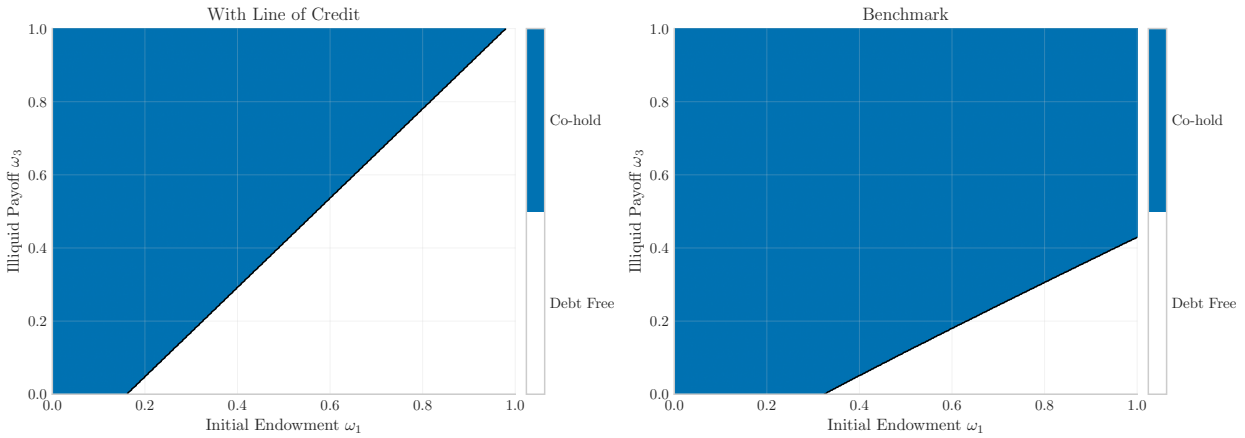
$$\theta_3 + u'(c_2) = \beta(1 + r_m)u'(c_3) + \phi. \quad (55)$$

In this case, the decision rules for non-borrowers ($d_{t+1} = 0$ and $a_{t+1} \geq 0$) is the same, $u'(c_1) = E[u'(c_2)]$ in the first period and $u'(c_2) = u'(c_3)$. Starting with a credit borrower in the second period, since $\beta(1+r_m)u'(c_3) < \beta(1+r_b)u'(c_3)$ it would have to be the case that $E[\phi] > 0$. Therefore, households will only use margins of credit if they exhaust their loan headroom and they become sufficiently constrained[16]. In the first period, households will not borrow against credit at all since the Euler equations would be inconsistent ($\beta(1+r_b)u'(c_2) > \beta(1+r_m)u'(c_2)$). This is because there is no upper bound on debt in the first period, so any household that needs liquidity now can take out a sufficiently sized loan.

The only difference between the baseline model and a model with credit margins is that it offers some flexibility for household second period liquidity needs. In other words,

15. While the borrowing limit never does bind, there is a weaker constraint at $a_{t+1} = 0$. That is because while the household can borrow, they need to be sufficiently constrained at $a_{t+1} = 0$ to be willing to pay the higher interest rate on credit.

16. We use the term sufficiently constrained here because the solution where $a_3 = 0$ is actually a corner solution. Because of the difference in rates, households will prefer not to borrow from credit unless their consumption drops too much despite making minimum loan payments.



(a) Consumer credit model (b) Benchmark model

Figure 10: Line of Credit Comparison

Note: These panels show the regions of co-holders ($a_2 > 0$, $d_2 > 0$) vs. asset holders ($a_2 > 0$, $d_2 = 0$) from the consumer credit model in panel (a) compared to the benchmark model in panel (b). Both models are parametrized by table 6, and take uniform expense shocks, $e \in (0, 0.4)$. The region in blue indicates co-holders. In simulation, the endowment space varies from 0.3 to 2 for both first period (horizontal axis) and third period (vertical axis) endowments. The endowments vary across the interval $(0.3, 2)$ and are normalized using the transformation $\hat{\omega} = (\omega - \underline{\omega})/(\bar{\omega} - \underline{\omega})$ so the axes read $(0, 1)$ for consistency.

co-holding is still optimal when future endowments are sufficiently large because closed-end loans are cheaper than consumer credit. However, since households anticipate second period liquidity needs conditional on expense draws, credit offers an additional insurance for unlucky households. This actually softens the blow from borrowing too little in the event of unfortunate expense shocks, which frees households to save on interest payments in the event of favorable shocks. Together, this means households keep less precautionary savings, and lower precautionary debt. The arguments are presented formally in proposition 4.

Proposition 4. *[Existence of Co-holding Solutions with Consumer Credit] Co-holding exists in the model with consumer credit. However, households choose weakly lower levels of debt when they have access to credit.*

Proof. See appendix D. □

Figure 10 compares decisions from the household and consumer credit models at the extensive margin. The result is consistent with proposition 4: Credit margins act as an insurance for households with unlucky draws. Figure 11 illustrates the second part of proposition 4, showing how much less co-holding households borrow at the intensive margin

The two most important predictions this extension finds is that despite the availability of consumer credit, households still choose to co-hold. However, the availability of credit, despite it being more expensive, may actually induce lower borrowing of cheaper closed-end

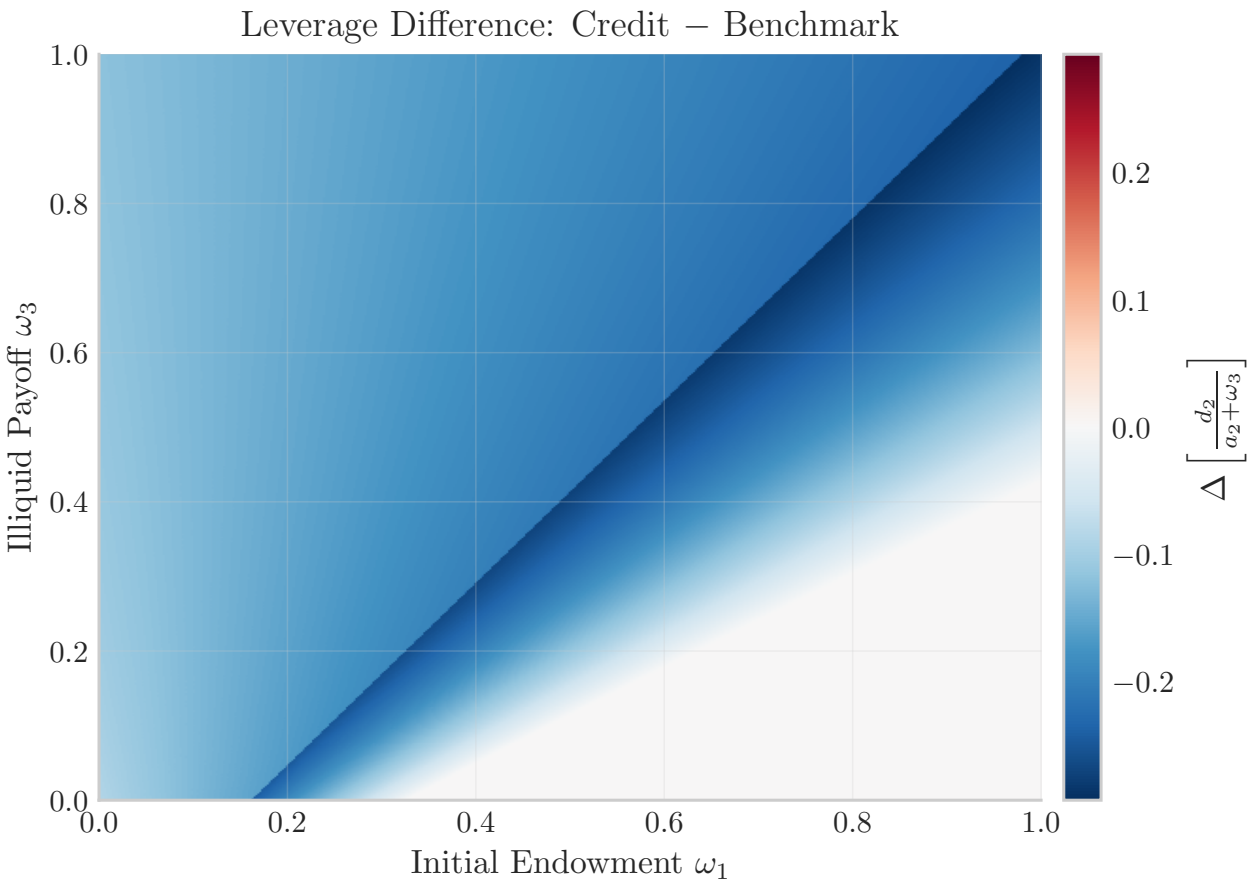


Figure 11: Leverage Choice Comparison

Note: This figure compares the difference in leverage choice from the benchmark model compared to the model in a heatmap. Brighter regions indicate the warmer regions. Leverage in each model is computed as the debt-to-asset ratio where assets include the first period asset choice and third period endowment. Both models are parametrized by table 6, and take uniform expense shocks, $e \in (0, 0.4)$. In simulation, the endowment space varies from 0.3 to 2 for both first period (horizontal axis) and third period (vertical axis) endowments. The endowments vary across the interval $(0.3, 2)$ and are normalized using the transformation $\hat{\omega} = (\omega - \underline{\omega})/(\bar{\omega} - \underline{\omega})$ so the axes read $(0, 1)$ for consistency.

loans. One interpretation of the interaction between closed-end loans and credit in this way is that households may use closed-end loans to satisfy long-term liquidity needs, while using credit to smooth short-term needs.

4.3 High Return Savings

Co-holding is only puzzling if in reality, the rate of return on household savings is less than the interest rate on debt. Otherwise, co-holding is a simple consequence of arbitrage. Thus, a natural objection of the mechanism developed in this paper is that co-holding is merely just a consequence of arbitrage. In other words, households are co-holding because they can use debt to finance investment in assets with returns that exceed the cost of debt. There are two versions of this objection. First, if the alternative asset has stochastic returns, the relevant phenomenon is leveraged investing, which standard portfolio theory predicts risk-averse households should engage in to some degree (Campbell 2006). This is a distinct behaviour that can coexist with the mechanism studied here, but the observed co-holders in section 2 do not look like leveraged investors: their liquid buffers are held in transaction accounts, not in risky assets earning a premium[17]. Bertaut, Haliassos, and Reiter (2009) and Telyukova (2013) reach similar conclusions in the credit-card setting. Second, if there is an alternative asset that has deterministic returns, proposition 5 establishes the conditions under which arbitrage is ruled out: so long as the return on the alternative asset is not too high and they have a different liquidity value than liquid assets, there is no arbitrage. This section extends the baseline model to show how the mechanism survives when households are given the option of investing in a higher deterministic return illiquid asset.

I will refer to the illiquid asset in this paper as a viscous asset, A, to emphasize that the asset is not fully liquid or illiquid: the idea that some assets may have lower short-term and higher long-run returns. The viscous asset will earn a return of $r_A > r_m > r$ each period, but must be held for two periods to earn r_A. For households that attempt to tap the asset in the second period, they will pay a proportional early withdrawal cost κ such that $(1-\kappa)r_A < r$. This latter assumption is necessary because if instead it was assumed that $(1-\kappa)r_A \geq r$, then there would be no incentive to purchase the liquid asset because any first period over-purchase of the viscous asset would be at least as valuable in the second period. Furthermore, since the viscous asset takes two periods to reach maturity, investment in the viscous asset will

17. Table 5 in particular notes the SCF LIQ variable which takes from checking, savings, money market, and call accounts

only happen in the first period. The new budget constraints are then

$$c_1 = \omega_1 + d_2 - a_2 - A_2 \tag{56}$$
$$c_2 = (1+r)a_2 + (1+(1-\kappa)r_A)(A_2 - A_3) - (1+r_m)d_2 - e - a_3 + d_3 \tag{57}$$
$$c_3 = (1+r)a_3 + (1+r_A)^2 A_3 - (1+r_m)d_3 + \omega_3. \tag{58}$$

and the full set of boundary constraints are

$$a_2 \geq 0,\ a_3 \geq 0,\ d_2 \geq 0,\ d_3 \geq 0,\ d_3 \leq \bar{d}(d_2),\ A_2 \geq 0,\ A_2 \geq A_3 \geq 0. \tag{59}$$

The main differences in the dynamic programming problem are that A_2 and A_3 are now state variables, $\boldsymbol{x}_2 = (a_2, A_2, d_2, e)$, $\boldsymbol{x}_3 = (a_3, A_3, d_3, \omega_3)$. It is hard to write a single dynamic program that captures the timing and constraints of each period without significantly increasing the notation so I write each time period's problem explicitly. In the first period

$$V(\omega_1) \equiv \max_{c_t, a_{t+1}, A_{t+1}, d_{t+1}} \Big\{ u(c_1) + \beta E[V(\boldsymbol{x}_2)] + \mu_1[\omega_1 - c_1 - a_2 + d_2] + \lambda_1 a_2 + \eta_1 A_2 + \theta_t d_{t+1} \Big\} \tag{60}$$

where η_1 captures the non-negativity of A_2. Since $(1+(1-\kappa)r_A) < (1+r_m)$ households would otherwise want to borrow from the asset. In the second period

$$\begin{aligned} V(\boldsymbol{x}_2) \equiv \max_{c_2, a_3, A_3, d_3} \Big\{ & u(c_2) + \beta E[V(\boldsymbol{x}_3)] + \\ & \mu_2[(1+r)a_2 + (1+(1-\kappa)r_A)(A_2 - A_3) - (1+r_m)d_2 - e - c_2 - a_3 + d_3] + \\ & \lambda_2 a_3 + \eta_2 A_3 + \psi_2 (A_2 - A_3) + \theta_2 d_3 + \phi \left(\bar{d}(d_t) - d_{t+1} \right) \Big\} \end{aligned} \tag{61}$$

where ψ captures the upper bound on early withdrawal of the viscous asset. In the final period, households consume the remainder of their net-worth so

$$V(\boldsymbol{x}_3) \equiv u(\omega_3 + (1+r)a_3 + (1+r_A)^2 A_3 - (1+r_m)d_3). \tag{62}$$

The question of interest in this model is whether households have an incentive to take on more debt to hold viscous assets. In other words, is there a profitable arbitrage in this model?

Fortunately, since the derivatives of the value function with respect to both a_{t+1} and d_{t+1} are independent of A_{t+1}, the Euler equations with respect to liquid assets and debt are unchanged. Consequently, the relationships between co-holding debt and liquid assets remains the same. The first-order and envelope theorem conditions with respect to the viscous asset

are

$$\mu_1 = \beta E_1[V_A(\boldsymbol{x}_2)] + \eta_1 \tag{63}$$
$$V_A(\boldsymbol{x}_2) = \mu_2(1 + (1-\kappa)r_A) + \psi_2 \tag{64}$$
$$\mu_2(1 + (1-\kappa)r_A) + \psi_2 = \beta V_A(\boldsymbol{x}_3) + \eta_2 \tag{65}$$
$$V_A(\boldsymbol{x}_3) = (1 + r_A)^2 u'(c_3). \tag{66}$$

Re-arranging, the full sets of Euler equations for the first period can be written as

$$u'(c_1) = E[u'(c_2)] + \lambda_1 \tag{67}$$
$$u'(c_1) = \beta(1 + (1-\kappa)r_A)E[u'(c_2)] + \beta E[\psi] + \eta_1 \tag{68}$$
$$\theta_1 + \beta E[\phi]\bar{d}'(d_2) + u'(c_1) = \beta(1 + r_m)E[u'(c_2)] \tag{69}$$

and in the second period

$$u'(c_2) = u'(c_3) + \lambda_2 \tag{70}$$
$$\psi + (1 + (1-\kappa)r_A)u'(c_2) = \beta(1 + r_A)^2 u'(c_3) + \eta_2 \tag{71}$$
$$\theta_2 + u'(c_2) = \beta(1 + r_m)u'(c_3) + \phi. \tag{72}$$

To study behaviour, there's a convenient relationship between first and third period consumption. Taking the first period expectation of equation 71, from equation 68

$$u'(c_1) = \beta^2(1 + r_A)^2 E[u'(c_3)] + \eta_1 + \beta E[\eta_2] \tag{73}$$

Thus, any solution that optimally requires $A_2 > 0$ and $E[\psi] > 0$ will yield

$$\frac{u'(c_1)}{E[u'(c_3)]} = \beta^2(1 + r_A)^2. \tag{74}$$

Furthermore, since $1 + r_A > 1 + r > 1 + (1-\kappa)r_A$, the second period return on liquid assets dominates the return on viscous assets and the third period return on viscous assets dominates the return on liquid assets. Therefore, in a deterministic setting, a household making optimal savings decisions would always choose just enough liquid assets to consume in the second period, and just enough viscous assets to consume in the third period. For this reason, $\lambda_1 = 0$ for all solutions implying $u'(c_1) = E[u'(c_2)]$ still holds. As such, any solution requiring $A_2 > 0$ will also require $a_2 > 0$.

Now, to see if households have an incentive to take on debt to increase A_2 savings in

the first period, first, I solve the model as if the loan market was initially shut down and see if access to loans can make households better off. Since a_2 is saved primarily for period 2 consumption and A_2 is for period 3 consumption, there is profitable arbitrage if r_A is significantly large. In fact it's not hard to see that if $r = r_m = 0$ and r_A is quite large, then since debt doesn't cost anything, households could easily just take on debt indefinitely to increase A_2. However, because households are required to make minimum payments on debt, it is possible to set conditions where no profitable arbitrage exists even with $r < r_m < r_A$. This is derived in Proposition 5

Proposition 5. *[No Arbitrage Between Debt and Viscous Assets] There is no arbitrage opportunity when* $0 < r < r_m < r_A$ *if and only if*

$$(1 + r_A)^2 \leq \frac{(1 + r)(1 + r_m)^2}{(2 + r_m)(1 + r) - (1 + r_m)^2}$$

Proof. See Appendix D. □

Using the values for r and r_m from table 6, no profitable arbitrage would exist in the example so long as

$$r_A < \sqrt{\frac{(1 + r)(1 + r_m)^2}{(2 + r_m)(1 + r) - (1 + r_m)^2}} - 1 \approx 0.13.$$

Proposition 5 rules out using debt purely to fund the viscous asset, but it does not rule out a three-way solution in which $a_2 > 0$, $A_2 > 0$ and $d_2 > 0$ hold together. It does, however, restrict when such a solution can occur. With $\beta(1 + r) = 1$, the bound in Proposition 5 can be written as

$$\beta^2(1 + r_A)^2 \leq \frac{\beta^2(1 + r_m)^2}{\beta(1 + r_m) - (\beta(1 + r_m) - 1)(2 + r_m)}. \tag{75}$$

Intuitively, this means the return on the viscous asset, measured as a ratio of period 1 to period 3 marginal utilities, cannot exceed the rate at which co-holding transfers resources from period 3 to period 1.

Proposition 6. *[Three-way co-holding solutions] Suppose the bound in Proposition 5 holds strictly and the viscous asset is held to period 3.*

1. *Without expense risk, no household holds* $a_2 > 0$, $A_2 > 0$ *and* $d_2 > 0$ *simultaneously.*

2. *With expense risk, a solution with* $a_2 > 0$, $A_2 > 0$ *and* $d_2 > 0$ *requires that the household be debt constrained in period 2 with probability strictly between zero and one.*

Proof. See Appendix D. □

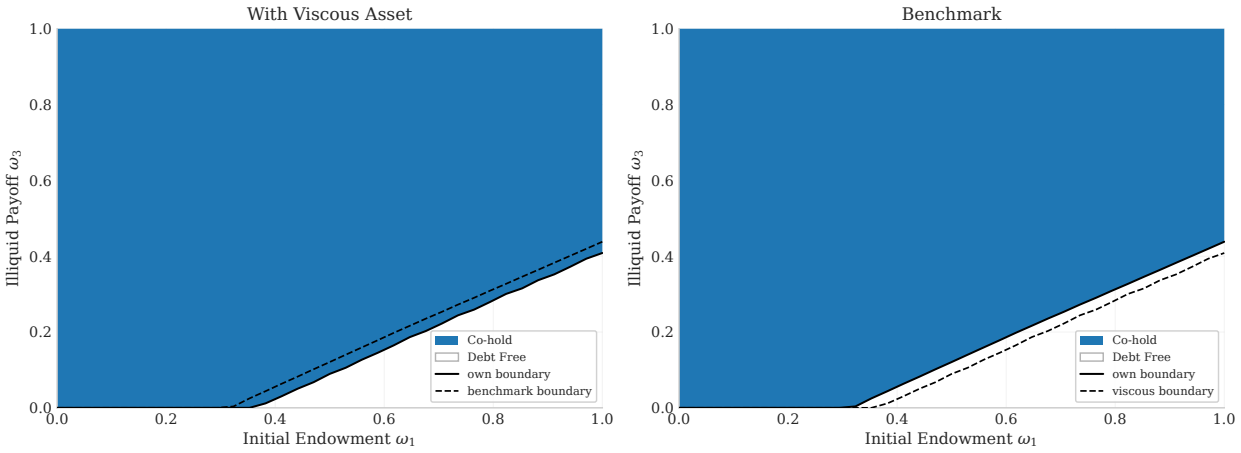
Proposition 6 says that the two objects answer different needs, and that risk is what creates both needs at once. Debt is valuable in the states where the household is debt constrained and the viscous asset is valuable in the states where it is not. Without risk, the household can either be constrained or unconstrained, so it will either co-hold liquid assets with debt or hold both assets, but never the three positions together. This is because in the deterministic setting, the ratio $u'(c_1)/u'(c_3)$ satisfies Proposition 1, and so $u'(c_1)/u'(c_3) > \beta^2(1+r_A)^2$ meaning the co-holder would rather consume in period 1 than invest in A_2. Under expense risk, the condition for A_2 involves the average ratio $E[u'(c_2)]/E[u'(c_3)]$, which the unconstrained states pulls below the Proposition 1 ratio, while the condition for debt involves only the constrained states. Because this involves the averages of being in both constrained and unconstrained states, both can hold together.

A numerical example is provided to show how introducing the viscous assets affects behaviour compared to the baseline. Table 7 shows the specific parameter choices used.

Table 7: Parameter choices for three object model

	Symbol	Value
Risk-free savings rate	r	0.05
Loan (debt) interest rate	r_m	0.10
Viscous asset rate	r_A	0.12
Early withdrawal penalty	κ	0.65
Discount factor	β	$1/(1+r)$

Figure 12 shows how the introduction of a viscous asset doesn't have much of an effect on co-holding overall. The reason for this is quite simple: co-holding is stronger for households who anticipate high period 3 endowments and therefore have very little incentive to increase A_2. However, because of the potential to increase period three consumption with good expense draws, households do find it optimal to carry at least some viscous assets. For this reason, households on the co-holding margin without the viscous asset have more incentive to hedge their bets and hold $A_2 > 0$, which leads to slight increases in d_2 and a_2 holdings as well. A_2 holdings are reported in table 7. Unsurprisingly, households with the largest ω_1/ω_3 hold the most significant viscous asset shares. However, there are contingents of households with weaker ratios that still find hedging $A_2 > 0$ despite being in the co-holding regions from table 12.



(a) Viscous asset model

(b) Benchmark model

Figure 12: Viscous Asset Comparison

Note: These panels show the regions of co-holders ($a_2 > 0$, $d_2 > 0$) vs. asset holders ($a_2 > 0$, $d_2 = 0$) from the viscous asset model in panel (a) compared to the benchmark model in panel (b). Both models are parametrized by table 7, and take uniform expense shocks, $e \in (0, 0.4)$. The region in blue indicates co-holders. The dashed overlay lines in both plots shows the difference in the regions from the other model. In simulation, the endowment space varies from 0.3 to 2 for both first period (horizontal axis) and third period (vertical axis) endowments. The endowments vary across the interval $(0.3, 2)$ and are normalized using the transformation $\hat{\omega} = (\omega - \underline{\omega})/(\bar{\omega} - \underline{\omega})$ so the axes read $(0, 1)$ for consistency.

4.3.1 Change in Early Withdrawal Fee

Given that $r_A > r_m > r$ and $(1 - \kappa)r_A < r$, the viscous asset has similar properties to US retirement saving accounts like IRAs and 401(k)s. Given that these accounts are tax advantaged, r_A maps nicely into returns and κ maps directly into early withdrawal penalties. For this reason, the model is well positioned to answer questions about changes in US retirement saving account policies. In particular, what does the model predict would happen if there was a change in the early withdrawal penalty κ? In a standard net-worth model the answer would be that savings increase, so this would be a trivial forum to evaluate the policy. Models that feature liquid and illiquid assets are better positioned to answer this question yet, they often still ignore the role of gross debt.

A lower early withdrawal penalty raises what the viscous asset pays in the states in which the household draws on it early, and leaves the payoff unchanged in the states in which it is held to period 3. This has two effects that work on opposite directions. First, the viscous asset becomes a better buffer since it still earns r_A if held to maturity but now also costs less to tap after realizing large expenses, so the household moves part of its precautionary liquidity into A_2. This is a substitution effect. Second, in period 2, households that withdraw early are wealthier in the counterfactual, so each extra dollar kept from period 2 to 3 is

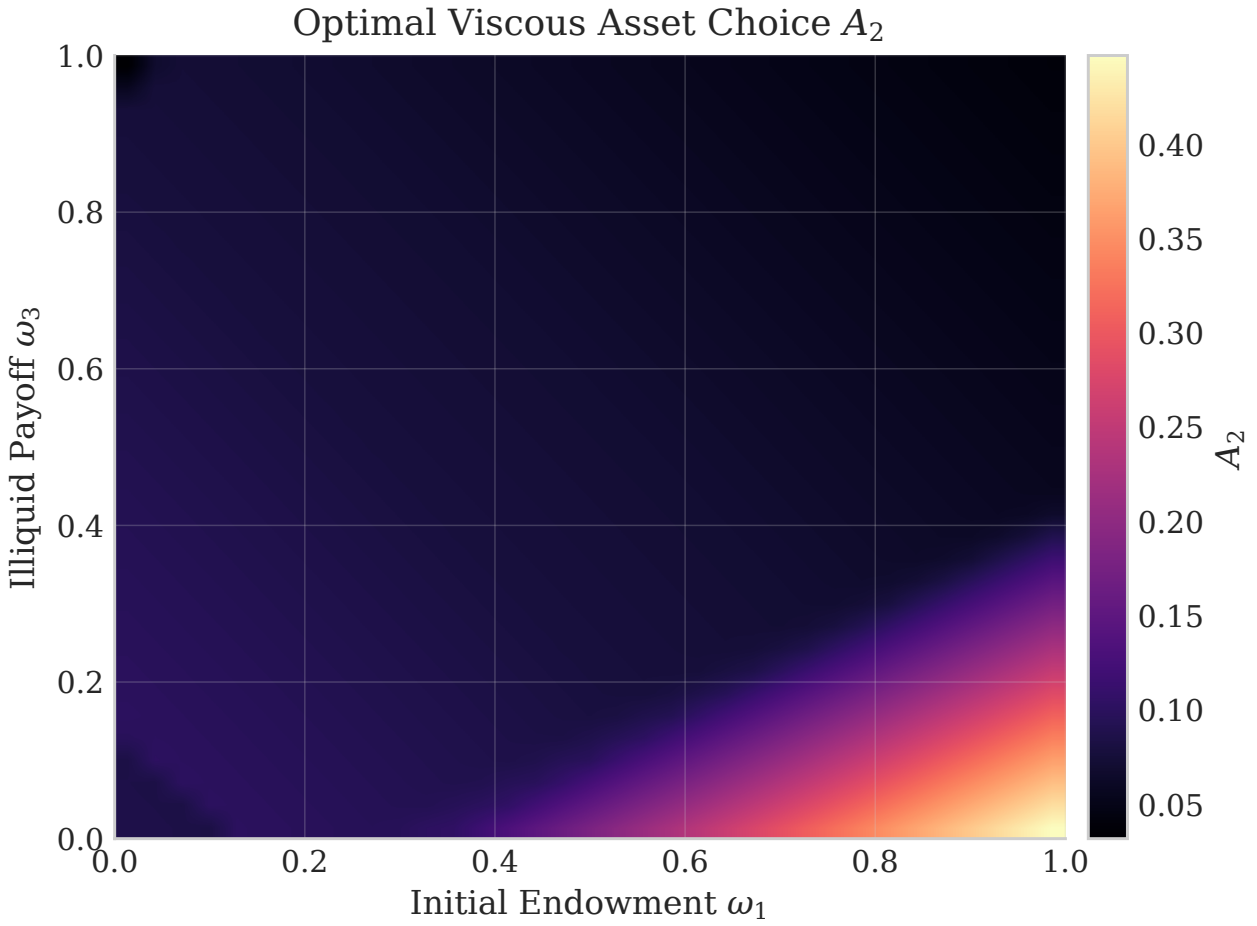


Figure 13: Viscous Asset Holdings

Note: This panel shows a heatmap of the viscous asset choice, A_2, in the first period plotted across the endowment space. The model is parametrized by table 7, and takes uniform expense shocks, $e \in (0, 0.4)$. Warmer regions are indicated by brighter colours. In simulation, the endowment space varies from 0.3 to 2 for both first period (horizontal axis) and third period (vertical axis) endowments. The endowments vary across the interval $(0.3, 2)$ and are normalized using the transformation $\hat{\omega} = (\omega - \underline{\omega})/(\bar{\omega} - \underline{\omega})$ so the axes read $(0, 1)$ for consistency.

worth less. This is an income effect. Unless households are relying on early withdrawals to buffer period 2 consumption, the substitution effect dominates. Generally, this would imply an increase in A_2 holdings resulting from a transfer of stored savings from a_2, and an ambiguous effect on d_2. Conversely, if the income effect dominates, there should still be a savings transfer from liquid to viscous assets, but because A_2 is used as a liquid buffer, there is likely to be an increase in d_2 because the premium on period 2 liquidity is cheaper.

The discussion over which effect dominates is especially relevant when comparing a netted asset model to the gross debt baseline. Because households in the netted model can take on negative asset positions, just like co-holders, there is an incentive to take on negative asset positions to hold more viscous assets. This means in net-asset models, the income effect is more likely to dominate.

To assess whether the distinction between gross and net positions matters for policy, I compare the welfare effect of an decrease in κ (lower early withdrawal fees) with the full model compared to one where the liquid asset is a net debt position[18]. The full model carries three portfolio objects: liquid savings, the illiquid asset, and debt. Welfare effects are reported as consumption-equivalent gains. Figure 14 plots the difference in the measured welfare gain across the endowment space. Two features stand out. First, the difference is positive throughout the region where the full model predicts co-holding, reaching near 1% of consumption. Second, it falls to zero in the high relative current endowment corner, which is precisely the region in which the full model predicts no debt is held, and thus where the two models coincide.

Welfare gains are stronger in the net worth model because the lack of an explicit debt margin forces households to a corner. Netted households with higher future endowments relative to current endowments smooth consumption by borrowing against their liquid asset position $a_2 < 0$ and save the additional proceeds in A_2. In the long-run, savings in A_2 provide better third period returns but the self-insure using the lower return second period asset because $a_2 < 0$ is the only form of precautionary borrowing. Households in the consolidated asset model save on average 8.5 times more in A_2 than in the full model using the parametrization in table 7. Therefore, reducing the second period cost of accessing A_2 benefits netted households more because of their higher choices in A_2, and ignoring debt explicitly overstates the true welfare gains.

While the results here appear conclusive, there is an important limitation worth highlighting. Neither of these models is calibrated to data and they are both computed with the same parametrization. Yet, any researcher asking quantitative welfare questions will calibrate their

18. Specifically, what I have in mind is that debt and assets are now a net position, such that $a_{net} = a - d$, and to preserve the same structure, if $a < 0$, the household is charged r_m.

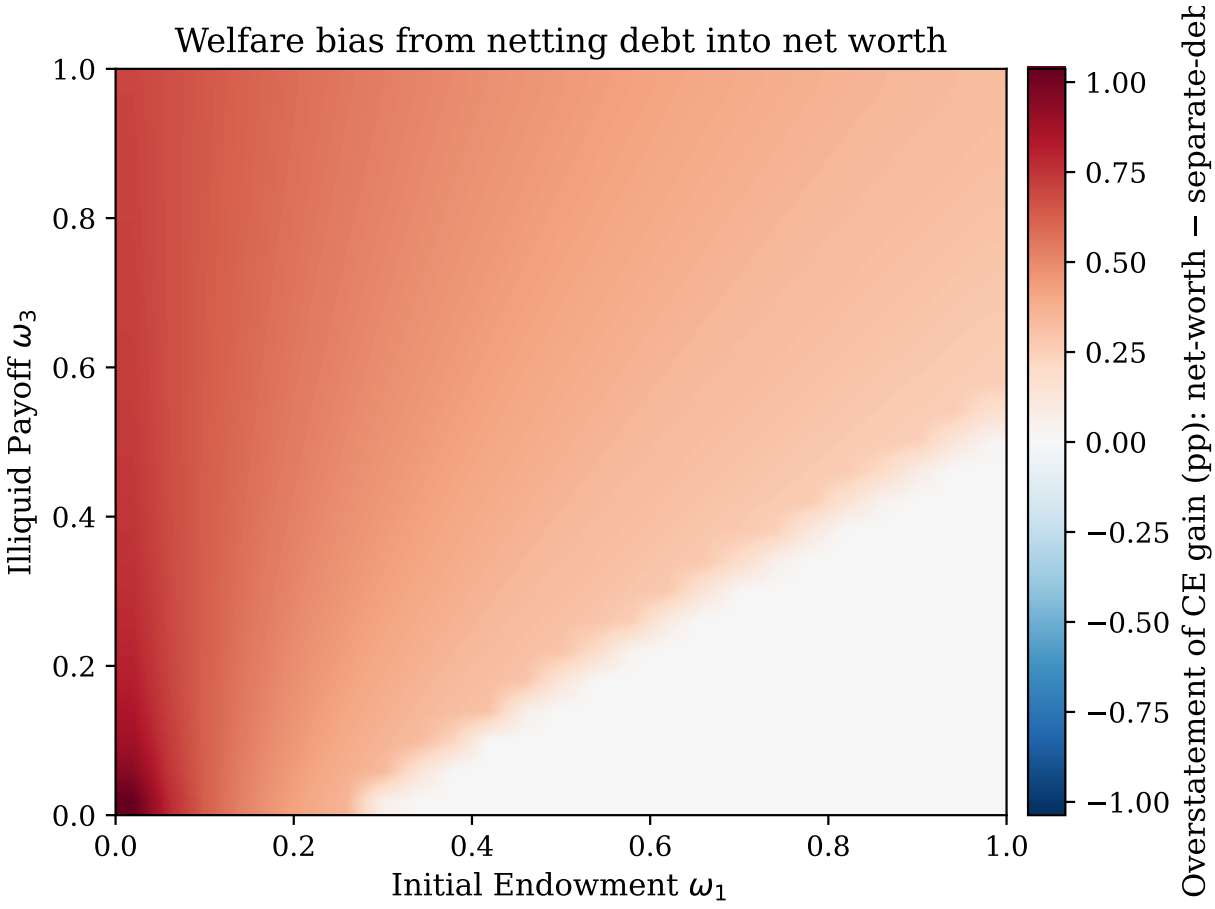


Figure 14: Comparison of welfare gains

Note: This panel shows a heatmap difference in consumption equivalent variation (CEV) due to a decrease in early withdrawal penalty, κ in a consolidated net asset model (e.g., $\tilde{a} = a - d$) compared to the viscous asset model. Warmer regions are indicated by brighter colours and show where the consolidated model reveals higher CEV compared to the benchmark.

models and therefore end up with different parameterizations, depending which model they use. Without resorting to any calibration, the cleanest robustness test I can run is to vary the parameterizations in both models simultaneously, and compare the results. To this end, store results from both models across variations in β and r_m. Regardless of parametrization, the consolidated model always has weakly higher CEV across the space[19].

5 Empirical Evidence of the Mechanism

This section provides suggestive empirical evidence on the relevance of the mechanism developed in the model. The objective is not to establish causal identification, but rather to examine whether the data exhibit patterns consistent with the theory's main comparative statics. In particular, the model predicts that co-holding should be more likely among households whose longer-run resources are strong relative to current cash flow, and that this behaviour should be especially relevant for less flexible, closed-end forms of debt. I investigate these predictions using reduced-form regressions in repeated cross-sections of household survey data. Because the theoretical objects of interest are not directly observed, the empirical analysis relies on proxy measures and should therefore be interpreted with caution. The results are best viewed as supportive evidence on whether the mechanism appears to be empirically relevant, rather than as a definitive test of the model.

The model predicts that households with larger ratios are more likely to co-hold (see Figure 9). Empirically, it is difficult to directly measure future endowments relative to current endowments. One possibility would be to use illiquid asset holdings, but this approach is imperfect. In particular, some illiquid assets, such as housing, are closely tied to broader household characteristics and debt choices, and may therefore reflect forces outside the mechanism emphasized in the model. In addition, although such assets are illiquid, they are not necessarily inaccessible over the horizon relevant for a simple three-period framework. For these reasons, measures based on future resources relative to current cash flow appear more suitable. Since the SCF does not provide panel data on household income dynamics, I instead construct proxy measures using variables available in the SCF summary extract, including current income, normal income, retirement wealth, and downside income shortfalls.

The empirical analysis focuses on co-holding in installment debt. I estimate weighted linear probability models in repeated SCF cross-sections for households ages 25 to 60 (see appendix B for the full specifications). The dependent variable is an indicator for simultaneously holding positive installment debt and a minimum amount of liquid assets. To assess robustness, I consider three definitions of liquidity holdings: any positive liquid assets, liquid assets

19. The routine for the robustness exercise is reported in appendix A

exceeding three months of income, and liquid assets exceeding twelve months of income. All specifications include controls for current income, age, age squared, marital status, wave fixed effects, and education fixed effects. Standard errors account for both the SCF replicate-weight design and the multiple-imputation structure.

The first and primary proxy is the gap between log normal income and log current income. Normal income is intended to capture a household's income in a typical year, while current income reflects realized income in the survey year. A household with higher normal income relative to current income may therefore be interpreted as having stronger longer-run resources relative to current cash flow. Table 8 reports the results. The main empirical finding is that this income-gap proxy is positive and statistically significant across all three definitions of co-holding. Moreover, the estimated relationship is stronger when co-holding is defined using more substantial liquidity buffers rather than merely positive liquid asset holdings. This pattern is consistent with the model's prediction that households with relatively stronger future resources are more likely to maintain liquidity while carrying closed-end debt.

Table 8: Normal-current income gap and co-holding in installment debt

	(1) LIQ > 0	(2) LIQ > 3 months income	(3) LIQ > 12 months income
Income gap	0.0157***	0.0287***	0.0192***
	(0.0040)	(0.0034)	(0.0032)
Log income	0.0317***	-0.0018	-0.0150***
	(0.0032)	(0.0027)	(0.0024)
Age	-0.0046**	-0.0012	0.0004
	(0.0023)	(0.0013)	(0.0004)
Married	-0.1257***	-0.0275***	-0.0160***
	(0.0053)	(0.0032)	(0.0020)
Wave FE	Yes	Yes	Yes
Education FE	Yes	Yes	Yes
Debt type	Installment	Installment	Installment
Observations	33,635	33,635	33,635
R^2	0.0714	0.0351	0.0675

Notes: Standard errors in parentheses. SCF-style standard errors combine replicate-weight sampling variation from implicate 1 with multiple-imputation variation across implicates.

The coefficient on current income displays a different pattern across columns. Under the loosest definition of co-holding, higher current income is associated with a greater likelihood of holding both debt and some liquid assets, which is unsurprising. Once the liquidity threshold is tightened, however, the coefficient becomes small or negative. This is also intuitive, since higher current income raises the amount of liquid assets required to satisfy income-based

buffer definitions such as three or twelve months of income. For this reason, current income is best interpreted here as a control rather than as a direct empirical analogue of the model's endowment ratio or first period endowment.

Table 9 introduces an additional proxy for longer-run resources by including retirement wealth relative to current income. Retirement wealth is measured in the SCF as holdings in retirement accounts, Keoghs, and pension-related accounts. Because these assets are largely illiquid over short horizons, they provide a useful proxy for resources that are available in the future but not readily accessible in the present. The coefficient on the retirement-wealth gap is positive and statistically significant across all specifications. Importantly, including this variable does not eliminate the positive relationship between the income-gap proxy and co-holding. Taken together, these results suggest that the baseline pattern is not driven solely by one particular proxy, but appears across multiple measures intended to capture stronger longer-run resources relative to current cash flow.

Table 9: Future-resource proxies and co-holding in installment debt

	(1) LIQ > 0	(2) LIQ $>$ 3 months income	(3) LIQ $>$ 12 months income
Income gap	0.0102**	0.0225***	0.0170***
	(0.0040)	(0.0034)	(0.0032)
Retirement gap	0.0041***	0.0047***	0.0016***
	(0.0005)	(0.0003)	(0.0002)
Log income	0.0276***	-0.0065**	-0.0167***
	(0.0032)	(0.0027)	(0.0025)
Age	-0.0052**	-0.0018	0.0002
	(0.0023)	(0.0013)	(0.0004)
Married	-0.1224***	-0.0238***	-0.0147***
	(0.0053)	(0.0031)	(0.0019)
Wave FE	Yes	Yes	Yes
Education FE	Yes	Yes	Yes
Debt type	Installment	Installment	Installment
Observations	33,635	33,635	33,635
R^2	0.0728	0.0414	0.0718

Notes: Standard errors in parentheses. SCF-style standard errors combine replicate-weight sampling variation from implicate 1 with multiple-imputation variation across implicates.

The regression reported in table 10 considers a more asymmetric income-based measure by replacing the baseline income gap with a downside income gap. This variable retains only positive values of the normal-current income gap and therefore isolates cases in which current income falls below normal income. The results are again positive and statistically

significant, and the coefficients remain strongest for the tighter liquidity-buffer definitions. This is broadly consistent with the idea that co-holding is especially relevant when current resources are temporarily weak relative to longer-run resources. The same regression also adds a children variable as a proxy for future expenditure needs. In contrast to the income-gap and retirement-gap variables, the estimated effect of children is not robust across the stricter co-holding definitions. This suggests that, once the main resource proxies are included, the children measure does not add much explanatory power in this setting. For that reason, it is best interpreted as a secondary and relatively weak proxy rather than as a central part of the empirical argument.

Table 10: Income shortfalls, retirement wealth, children, and co-holding

	(1) LIQ > 0	(2) LIQ $>$ 3 months income	(3) LIQ $>$ 12 months income
Downside income gap	0.0088**	0.0254***	0.0211***
	(0.0042)	(0.0037)	(0.0035)
Retirement gap	0.0044***	0.0046***	0.0016***
	(0.0005)	(0.0003)	(0.0002)
Children	0.0183***	-0.0017	0.0003
	(0.0021)	(0.0012)	(0.0005)
Log income	0.1246***	-0.0083	-0.0266***
	(0.0105)	(0.0072)	(0.0068)
Log income $\times$ age	-0.0022***	0.0001	0.0003*
	(0.0002)	(0.0002)	(0.0001)
Age	0.0114***	-0.0020	-0.0025
	(0.0030)	(0.0020)	(0.0015)
Married	-0.1098***	-0.0245***	-0.0138***
	(0.0055)	(0.0032)	(0.0019)
Wave FE	Yes	Yes	Yes
Education FE	Yes	Yes	Yes
Debt type	Installment	Installment	Installment
Observations	33,635	33,635	33,635
R^2	0.0774	0.0418	0.0755

Notes: Standard errors in parentheses. SCF-style standard errors combine replicate-weight sampling variation from implicate 1 with multiple-imputation variation across implicates.

In addition to the prediction that co-holding is more likely when future endowments are relatively large, the model also comments risk could lead to higher co-holding and leverage. Since the SCF is a time-series cross-section and not a panel, this is particularly difficult to proxy for. Each table reports marriage as a control, which is negative and statistically significant across all specifications and co-holding definitions. One view is that married

households may face less effective income risk, since access to two potential earners can provide some insurance against adverse income shocks. If true, then the negative coefficient could suggest that co-holding is more prevalent when households face greater risk. This interpretation, however, should be treated cautiously. Marriage is likely correlated with many other household characteristics, including household composition, expenditure needs, debt structure, and financial organization, so the estimated coefficient should not be interpreted as a clean measure of income risk alone.

Overall, the reduced-form evidence is broadly consistent with the model's main predictions. Households with stronger future resources relative to current cash flow, as measured by normal income and retirement wealth proxies, appear more likely to co-hold installment debt and liquidity. This relationship is strongest when co-holding is defined using more meaningful liquidity buffers rather than merely positive liquid balances. However, none of these results should be interpreted as causal evidence since each proxy is an imperfect measure of the model's objects. Since causal inference is outside of the scope of this paper, future work should focus on testing the model's predictions from panels where more robust measures of endowments and risk can be defined.

6 Conclusion

The rational co-holding literature has been quite successful in explaining the credit card debt puzzle. However, the co-holding puzzle also exists for mortgages and other installment loans, and due to the different structure, cannot be explained through the same mechanisms. To fill the gap, this paper develops a simple theory that embeds the strict closed-end nature and repayment schedules of installment loans. Liquid assets and closed-end debt coexist in portfolios because households can use loans to tap future liquidity and self-insure against idiosyncratic risk. Even with extensions that allow for access to consumer credit and higher return illiquid assets, co-holding remains optimal so long as the interest on credit exceeds the interest on loans, and the spreads between rates aren't excessively large. Models that net out debt positions may also misstate gains from policy because these models ignore both the option value that co-holding provides, and the cost of precautionary debt. The theoretical predictions from this paper are consistent with patterns in the SCF. In particular, households are more likely to co-hold when current income is lower than expected future income, and with relatively higher retirement savings.

The portfolio also matters for evaluating policy. In the deterministic version of the model, net worth is a sufficient statistic for welfare. Once expense shocks are introduced it no longer is: households with the same net worth hold different portfolios in the second period, and the

more leveraged among them hold more liquid assets and are less exposed to the shock. In the viscous-asset extension, this changes measured consumption-equivalent variations, because gross asset and debt positions give households a margin for precautionary holdings that a consolidated model does not have.

The theory also has some important limitations: the model only features three periods, and installment loans are chosen in the first period, leaving households no choice to refinance later. The framework therefore fits best a precautionary debt story in which households borrow under favorable current conditions in anticipation of future liquidity needs. A companion paper, *Debt as Insurance*, relaxes both limitations in a life-cycle model of retirement in which households refinance at a cost and may lose access to credit with age, health, and income. For retirees, whose refinancing opportunities are limited, and whose main asset is the home, the closed-end structure of mortgage debt is especially relevant. The mechanism developed here, borrowing under favourable conditions against future liquidity needs, is what the estimated model finds behind co-holding in the HRS.

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A Data

This appendix describes the methodology used to collect evidence from the Survey of Consumer Finances (SCF) in sections 2 and 5.

A.1 Source and Sample

Data in this paper is sourced primarily from the Survey of Consumer Finances summary extract files using all 12 waves from 1989-2022. Section 2 uses the full sample ($N = 58,148$) and 5 restricts the sample to ages 25-60 ($N = 33,635$) and the waves to 1995-2022 since some variables used did not exist prior to. Table 11 reports the summary of the entire sample weighted by the survey weights (sample counts are unweighted).

A.2 Variable definitions

Co-holding in the data is defined at the individual level using the following relation

$$I_{\text{co-holder}} = \begin{cases} 1 \text{ if } a > \underline{a} \text{ and } d > 0 \\ 0 \text{ otherwise} \end{cases} \tag{76}$$

where a is liquid assets, $\underline{a}$ is the asset threshold, and d is debt. In general, the asset threshold, $\underline{a}$, is set equal to three months income unless otherwise specified. Specific definitions of liquid assets and debt used are listed in table 12.

Interest rates on loans are taken directly from the raw SCF data files. To insure the carry costs in equation 1 are not overstated, loan balances are only used if the interest rate is reported, otherwise it is dropped from the calculation. Returns on deposits are taken from FRED sourced from the FDIC national rates, with the interest rates on checking accounts assumed to be zero. Since the FDIC changed its definition of deposit rates in April 2021, the sources are different for waves prior to 2022. Returns on savings accounts are estimated using the FRED SNDR series for the 2022 wave and SAVNRNJ prior to. Money market account (MMA) rates are inferred from MMNDR in 2022 and MMNRNJ for previous waves.

Table 11: SCF analysis sample

Panel A: sample by wave

		Medians (2022 $k)		Holds debt (%)			Co-hold, 3mo (%)	
Wave	HH	Income	Liquid	Mort.	Inst.	CC	Mort.	Inst.
1989	3,143	60.1	3.5	39.5	49.4	39.7	6.0	6.0
1992	3,906	55.2	3.4	39.1	46.0	43.7	7.0	5.4
1995	4,299	59.5	3.1	41.0	46.0	47.3	6.0	4.6
1998	4,305	61.0	4.6	43.1	43.8	44.1	8.4	6.1
2001	4,442	67.2	5.0	44.6	45.2	44.4	9.4	6.1
2004	4,519	67.8	4.7	47.9	46.0	46.2	9.9	6.7
2007	4,417	67.7	4.7	48.7	46.9	46.1	9.2	6.0
2010	6,482	62.5	3.8	47.0	46.4	39.4	9.1	6.0
2013	6,015	59.4	4.3	42.9	47.2	38.1	9.5	7.4
2016	6,248	65.0	5.1	41.9	50.2	43.9	10.1	8.9
2019	5,777	68.5	5.8	42.1	51.5	45.4	10.8	9.6
2022	4,595	70.3	7.8	42.2	53.1	45.2	13.5	12.4

Panel B: 2022 wave, weighted

	Mean	Median	p90	Share > 0 (%)
Income	141.4	70.3	248.6	99.6
Normal income	131.4	73.5	245.0	99.9
Liquid assets	61.6	7.9	115.3	98.6
Total debt	126.9	31.7	340.0	77.4
Mortgage debt	89.6	0.0	268.7	42.2
Installment debt	19.5	0.8	52.7	53.1
Credit-card balance	2.8	0.0	8.0	45.2
Age	52.4	52.0	76.0	100.0

Notes: Cleaned SCF summary-extract sample; each household enters as five implicates and HH counts unique households. Statistics use survey weights pooled across implicates. Dollar values are in thousands of 2022 dollars as published in the Board's inflation-adjusted summary extract (CPI-U adjustment to the latest survey year); chapter 3's HRS values are deflated to the same base year via FRED CPIAUCSL (Age in years). Co-holding is liquid assets of at least three months of income together with positive debt of the stated type (Section 2.2; taxonomy of Section 1.2.1).

Since the FDIC only tracks deposit rates, rates for call accounts are not directly observed. I assume that call accounts thus take on the same rate as MMAs for two reasons. First, call accounts are typically below MMA deposit rates, so the assumption likely understates carry costs here. Second, call accounts make up only a small fraction of the SCF LIQ variable used for liquid assets.

The carry cost calculation, equation 1, is computed by estimating portfolio marginal returns to liquid assets, r_a, and marginal interest on loans, r_b, and using the spread as the true portfolio cost of debt. Specifically, from the raw data files, for each household I construct $b_{j,h}$ as the balance of loan j for household h. If the household reports the yearly interest rate on that specific loan, $r_{j,h}$, then the balance and interest rate are used in equation 77, otherwise it is dropped. The marginal rate on debt is thus

$$r_{\text{DEBT},h} = \frac{r_{1,h}b_{1,h} + \cdots + r_{J,h}b_{J,h}}{b_{1,h} + \cdots + b_{J,h}}. \tag{77}$$

For liquid assets, the process is similar, just that returns are homogeneous by assumption given the data limitations. To be explicit, the return on liquid assets is given by

$$r_{LIQ,h} = \frac{r_{\text{SAVING}} \cdot \text{SAVING} + r_{\text{MMA}} \cdot (\text{CALL} + \text{MMA})}{\text{CHECKING} + \text{SAVING} + \text{CALL} + \text{MMA}}. \tag{78}$$

Table 12: Variable definitions, SCF summary extract

Object	SCF construction	Notes
Liquid assets (a)		
Liquid assets	LIQ = CHECKING + SAVING + MMA + CALL	Transaction, savings, money market, and call accounts. MMA bundles money-market deposit accounts with money-market mutual funds.
Debt (d), by type		
Total debt	DEBT	All household debt; used in the aggregate trend figures.
Mortgage debt	MRTHEL	Debt secured by the primary residence, including home-equity loans and HELOC balances, not first liens only.
Installment debt	INSTALL	Closed-end consumer loans (vehicle, education, other); the debt concept in the Section 5 regressions.
Credit-card debt	CCBAL	Balance remaining after the last statement payment, i.e. revolving balances only; transactor balances inside the grace period are excluded.
Income and retirement savings		
Income	INCOME	Total household income in the survey year; $\underline{a}$ = INCOME/4 (three months).
Normal income	NORMINC	Self-assessed income in a "normal" year; available from 1995, used in the Section 5 income-gap proxy.
Retirement savings	RETQLIQ	Sum of Individual Retirement Accounts, Keoghs, and Account-type Defined Contribution (DC) pensions (e.g., 401(k)s, 403(b)s, and thrift savings plan).

Note: Statistics computed from these variables are weighted by the SCF survey weights and pooled across the five implicates unless noted. Dollar values are in 2022 dollars as published in the Board's inflation-adjusted extract.

A.3 Measurement Conventions

Deflation and weighting. All dollar values are in 2022 dollars as published in the SCF board's inflation-adjusted summary extract (CPI-U-RS, to the latest survey year), no further deflation is applied. Descriptive statistics and cross-section measures in section 2 are weighted by the survey weights pooled across all 5 SCF implicates (this is equivalent to averaging across implicates)[20]. Regression standard errors combine replicate-weight sampling variation with multiple-imputation variable across implicates, the specific process is discussed in section B.

Interest rates. Rates are nominal throughout the carry-cost calculation, equations 1, 2, and 3. Since the carry cost depends on the spread between the rates, the definition of spread costs is invariant to real or nominal specification. Rates on debt are the SCF's own reported loan rates from the raw data files. Liquid returns are found from the FDIC's national deposit rates for the given survey year, and the rate on checking balances is assumed to be zero. The SCF summary extract's MMA bundles deposit accounts with money-market mutual funds, and hence using FDIC national deposit rates may overstate the results from equation 1. This motivates the robustness check where liquid assets are defined using checking and saving account balances only.

B Regressions

This appendix lists the regressions equations used for estimation in section 5.

Equation 79 is the specification used for the linear probability model estimated in table 8.

$$P(C_{i,t} = 1) = \beta_1 GAP_{i,t} + \beta_2 INCOME_{i,t} + \beta_3 AGE_{i,t} + \beta_4 AGE_{i,t}^2 + \boldsymbol{\alpha}'\boldsymbol{\tau}_{i,t} + \boldsymbol{\gamma}'\boldsymbol{x}_{i,t} + \varepsilon_{i,t} \quad (79)$$

where C denotes co-holding, $\boldsymbol{\alpha}$ is the vector coefficient of time fixed effects, $\boldsymbol{\gamma}$ is the vector coefficient of education fixed effects, and GAP refers to the log ratio

$$GAP = \ln \frac{NORMINC}{INCOME} \quad (80)$$

which measures the log difference of the SCF normal income variable to actual income. In the SCF summary extract, normal income refers to the amount of income a household would have earned in a normal year, income is the amount the household actually received. The GAP variable uses the log ratio of income. $\boldsymbol{\tau}$ is the vector of survey year fixed effects, and $\boldsymbol{x}$ is a vector of controls, controlling for highest grade of education, age, age squared, marriage,

20. Sample counts are unweighted unless otherwise specified.

and an intercept term. Highest grade, age, and age squared are all included to account for earnings profiles. Equations 81 and 82 are alternate specifications reported in tables 9 and 10 respectively.

$$P(C_{i,t}=1) = \beta_1 GAP_{i,t} + \beta_2 RETGAP_{i,t} + \beta_3 INCOME_{i,t} + \beta_4 AGE_{i,t} + \beta_5 AGE_{i,t}^2 + \boldsymbol{\alpha}'\boldsymbol{\tau}_{i,t} + \boldsymbol{\gamma}'\boldsymbol{x}_{i,t} + \varepsilon_{i,t} \quad (81)$$

$$P(C_{i,t}=1) = \beta_1 DGAP_{i,t} + \beta_2 RETGAP_{i,t} + \beta_3 CHILDREN_{i,t} + \beta_4 INCOME_{i,t} + \beta_5 AGE_{i,t} + \beta_6 AGE_{i,t}^2 + \beta_7 \log INCOME_{i,t} \times AGE_{i,t} + \boldsymbol{\alpha}'\boldsymbol{\tau}_{i,t} + \boldsymbol{\gamma}'\boldsymbol{x}_{i,t} + \varepsilon_{i,t} \quad (82)$$

Note that $RETGAP$ in equation 81 refers to the log ratio of $NORMINC$ to $RETQLIQ$ and $DGAP$ in equation 82 only considers income in a down year[21]. Estimation follows the SCF's implicate and replicate weight design. Each regression is estimated by weighted least squares, using WGT, the survey's household weights. Each regression is estimated once per implicate and the results report the cross implicate averages (Rubin 1987).

Since the SCF is a dual-frame design (area-probability sample plus an oversample of wealth households drawn from tax data), standard bootstrap resampling ignores this structure and would hence be invalid since disclosure rules mean that the geographic and frame identifiers needed to replicate survey design are undisclosed. For this reason, the SCF provides researchers with 999 replicate weights to produce bootstrap standard errors that are equivalent to design-consistent resampling. In the replicate weight files, the SCF provides a base weight (WT1Bj) and a multiplicity factor (MMj), how many times the case was actually drawn), calibrated against the first implicate. For each replicate I re-run the estimation procedure using the replicate weights

$$r_w = \max(0, WT1Bj) \times \max(0, MMj) \quad (83)$$

as the replicate sample weights and compute the variance

$$\hat{V}_r = \frac{1}{R-1} \sum_i^R \left(\hat{\theta}_r - \bar{\theta}\right)^2, \quad R = 999. \quad (84)$$

This however does not capture imputation uncertainty because it holds the implicate constant. To correct for this, I compute the total sample variance using Rubin's rule

$$\hat{V}_{\text{total}} = \hat{V}_r + \left(1 + \frac{1}{m}\right) \cdot B, \quad m = 5 \quad (85)$$

21. For households with normal income higher than realized income, GAP is set equal 0.

where B is the between-implicate variance

$$B = \frac{1}{m-1}\sum_{i}^{m}\left(\hat{\theta}_i^M - \bar{\theta}^M\right)^2. \tag{86}$$

C Numerical Solution Routines

Each of the numerical examples in the baseline model and extensions is solved directly using the relevant first-order conditions. Since each model is similar, here I simply outline the procedure for a general case that is applied to every model except the viscous asset extension. Since that model has three choice variables, I specify this routine separately.

C.1 Routine for sections 3 to 4.2

Given first period choices of a_2 and d_2, the second period problem can be separated into differed parts depending on the expense shock as follows:

$$\begin{cases} e \leq e' & \text{The household is a saver } (a_3 \geq 0, d_3 = 0) \\ e' < e \leq e^* & \text{The household neither saver nor borrower } (a_3 = d_3 = 0) \\ e \geq e^* & \text{The household is a borrower } (a_3 = 0, d_3 \geq 0). \end{cases} \tag{87}$$

Based on these thresholds, a_3 and d_3 can be written as the following functions using the first-order conditions and recalling that $u(c) = \ln c$

$$a_3(a_2, d_2, e) = \begin{cases} \frac{(1+r)a_2 - (1+r_m)d_2 - \omega_3 - e}{2+r} & \text{if } e < e' \\ 0 & \text{otherwise} \end{cases} \tag{88}$$

$$d_3(a_2, d_2, e) = \begin{cases} \min\left\{\frac{(1+r_m)d_2 - (1+r)a_2 + \omega_3 + e}{2+r_m}, \frac{1+r_m}{2+r_m}d_2\right\} & \text{if } e > e^* \\ 0 & \text{otherwise} \end{cases} \tag{89}$$

In the extension to lines of credit with $a < 0$, another threshold where borrowing becomes optimal can be defined at $e'' > e^*$ where $d_3 = \frac{1+r_m}{2+r_m}d_2$ and $a_3 < 0$ such that

$$a_3(a_2, d_2, e) = \begin{cases} \frac{(1+r)a_2 - (1+r_m)d_2 - \omega_3 - e}{2+r_b} & \text{if } e > e'' \\ 0 & \text{otherwise.} \end{cases} \tag{90}$$

Using these definitions, the problem can be solved as a first-period problem, taking expectations along the thresholds for e. Let the first-order conditions generally be given

as

$$\frac{1}{\omega_1 - a_2 + d_2} = E\left[\frac{\beta(1+R)}{(1+r)a_2 - (1+r_m)d_2 - a_3(a_2,d_2,e) + d_3(a_2,d_2,e) - e}\right] \tag{91}$$

$$\beta E[\phi]\bar{d}'(d_2) + \frac{1}{\omega_1 - a_2 + d_2} = E\left[\frac{\beta(1+r_m)}{(1+r)a_2 - (1+r_m)d_2 - a_3(a_2,d_2,e) + d_3(a_2,d_2,e) - e}\right] \tag{92}$$

$$\phi = \max\left\{\frac{1}{(1+r)a_2 - (1+r_m)d_2 - a_3(a_2,d_2,e) + d_3(a_2,d_2,e) - e} - \frac{1}{\omega_3 + (1+r)a_3(a_2,d_2,e) - (1+r_m)d_3(a_2,d_2,e)}, 0\right\} \tag{93}$$

The optimal choices are then solved using a nested bisection method, where the limits of integration in the expectation are defined from $(0, e'), (e', e^*), (e^*, e''), (e'', \bar{e})$. Each model is solved along a grid of starting points for ω_1 and ω_3.

C.2 Routines For Viscous Asset Model

The viscous asset model is solved using backward induction along a grid from the value function. The routine relies on nested golden section search and future value functions are linearly interpolated to estimate and store decision rules.

The CEV experiment considers how a consolidated asset net of debt model performs compared to the baseline. The consolidated model sets the net asset $\tilde{a} = a - d$. The three budget constraints in the consolidated model become

$$c_1 = \omega_1 - \tilde{a}_2 - A_2 \tag{94}$$

$$c_2 = (1+R)\tilde{a}_2 + (1 + (1-\kappa)r_A)(A_2 - A_3) - e - \tilde{a}_3 \tag{95}$$

$$c_3 = (1+R)\tilde{a}_3 + (1+r_A)^2 A_3 + \omega_3 \tag{96}$$

and the value functions can be written as

$$V(\omega_1) \equiv \max_{c_t, \tilde{a}_{t+1}, A_{t+1}} \left\{u(c_1) + \beta(1+R)E[V(\boldsymbol{x}_2)] + \mu_1[\omega_1 - c_1 - \tilde{a}_2] + \eta_1 A_2\right\} \tag{97}$$

$$V(\boldsymbol{x}_2) \equiv \max_{c_2,\tilde{a}_3,A_3} \Big\{ u(c_2) + \beta(1+R)E[V(\boldsymbol{x}_3)] + \mu_2[(1+R)\tilde{a}_2 + (1+(1-\kappa)r_A)A_2 - e - c_2 - \tilde{a}_3] + \lambda_2 a_3 + \psi_2 (A_2 - A_3) \Big\} \quad (98)$$

$$V(\boldsymbol{x}_3) \equiv u(\omega_3 + (1+R)\tilde{a}_3 + (1+r_A)^2 A_3). \quad (99)$$

where

$$\begin{cases} R = r & \text{if } \tilde{a}_2 \geq 0 \\ R = r_m & \text{if } \tilde{a}_2 < 0. \end{cases} \quad (100)$$

This model is solved in the same way as the baseline.

Consumption equivalent variations (CEV) are computed using the following procedure. First, the value functions for each case and each model are solved for $\kappa = 0.65$ and $\kappa = 0.60$ following

$$V_1(\boldsymbol{x}; \kappa) = \log c_1 + \beta \log c_2 + \beta^2 \log c_3. \quad (101)$$

For each model and each endowment pair, the CEV is the permanent proportional increase in consumption that would leave households indifferent from the baseline and reform counterfactual. Since I use log utility, scaling consumption by $1+g$ in each period would increase lifetime utility by the factor

$$(1 + \beta + \beta^2) \log(1+g). \quad (102)$$

The CEV for each model can therefore be computed as

$$g(\boldsymbol{x}) = \exp\left\{ \frac{V_1(\boldsymbol{x}, 0.60) - V_1(\boldsymbol{x}, 0.65)}{1 + \beta + \beta^2} \right\}. \quad (103)$$

The exercise then reports the difference between $g^{\text{consolidated}} - g^{\text{baseline}}$.

Because a differently specified model would have differently calibrated parameters, I run variations across different parameter choices for both models. The parameters are varied according to the following grids

$$\beta \in \{0.90, 0.945, 0.99\} \quad (104)$$

$$r_m \in \{0.08, 0.10, 0.14\} \quad (105)$$

$$\kappa^{\text{reform drop}} \in \{0.65 \to 0.60, 0.65 \to 0.62\}. \quad (106)$$

Both models are re-solved by permutation across all 8 parameter changes.

D Proofs

Proposition 1. *[Co-holding Solutions] The set of co-holding solutions is*

$$\mathbb{C} = \left[(c_1, c_2, c_3) \middle| c_1 = c_2 \ \& \ \frac{u'(c_2)}{u'(c_3)} = \frac{\beta^2(1+r_m)^2}{\beta(1+r_m) - (\beta(1+r_m)-1)(2+r_m)} \right].$$

Proof. Suppose that the household is a co-holder. The necessary conditions are then $\lambda_2 = 0$ and $\phi > 0$. For $\phi > 0$ to be true, it must then also be the case that $\theta_3 = 0$. Consequently, since there is no second period endowment, to avoid $c_2 = 0$, the household must always choose $a_2 > 0$. Therefore, any debtor is also a co-holder. Solving equation 20 for ϕ, we can write

$$\phi = \frac{(\beta(1+r_m)-1)(2+r_m)}{\beta(1+r_m)^2} u'(c_2)$$

where we've made the substitutions $u'(c_1) = u'(c_2)$ and

$$\bar{d}'(d_2) = 1 + r_m - \frac{r_m(1+r_m)^2}{(1+r_m)^2 - 1} = \frac{1+r_m}{2+r_m}.$$

Then, substituting the expression into equation 21 we can write

$$u'(c_2) = \beta(1+r_m)u'(c_3) + \frac{(\beta(1+r_m)-1)(2+r_m)}{\beta(1+r_m)} u'(c_2).$$

Finally, re-arranging, we obtain

$$\frac{u'(c_2)}{u'(c_3)} = \frac{\beta^2(1+r_m)^2}{\beta(1+r_m) - (\beta(1+r_m)-1)(2+r_m)}$$

□

Proposition 2. *[Initial Conditions that guarantee co-holding] A household will choose to co-hold if and only if*

$$\frac{u'\left(\frac{1+r}{2+r}\omega_1\right)}{u'(\omega_3)} > \frac{\beta^2(1+r_m)^2}{\beta(1+r_m) - (\beta(1+r_m)-1)(2+r_m)}$$

Proof. From proposition 1, the choice to co-hold depends on

$$\begin{aligned} c_1 &= \omega_1 + d_2 - a_2 \\ c_2 &= (1+r)a_2 - (1+r_m)d_2 + d_3 - a_3 \\ c_3 &= \omega_3 + (1+r)a_3 - (1+r_m)d_3. \end{aligned}$$

Households on the margin as $d_2 \geq 0$ just binds ($\phi = 0$ and $\theta_2 = \theta_3 = 0$) will choose $a_2 > 0$ and $d_2 = 0$ optimally. If ω_3/ω_1 increases any further, then $a_2 > 0$ and $d_2 > 0$ becomes optimal. From lemma 1, this also implies that $a_3 = 0$. Thus, the new consumption set becomes

$$\begin{aligned} c_1 &= \omega_1 - a_2 \\ c_2 &= (1+r)a_2 \\ c_3 &= \omega_3. \end{aligned}$$

Optimality requires that $c_1 = c_2$ and thus

$$a_2 = \frac{\omega_1}{2+r} \implies c_1 = \frac{\omega_1(1+r)}{2+r}.$$

and $c_3 = \omega_3$. Therefore, from proposition 1, if

$$\frac{u'\left(\frac{1+r}{2+r}\omega_1\right)}{u'(\omega_3)} > \frac{\beta^2(1+r_m)^2}{\beta(1+r_m) - (\beta(1+r_m)-1)(2+r_m)}$$

then it is optimal for the household to choose $d_2 > 0$. □

Proposition 3. *Let $\bar{e}$ be the upper bound of the support for $\mathcal{F}$. For a given choice of a_2, $d_2 > 0$ is optimal only if for some $e^* < \bar{e}$, the household is debt constrained in the second period.*

Proof. Suppose the household chooses to save at the level a_2. In the second period, the household is debt constrained if and only if $u'(c_2) > \beta(1+r_m)u'(c_3) \iff \phi > 0$, which in turn implies that $u'(c_2) > u'(c_3) \iff \lambda_3 > 0$ since $\beta(1+r_m) > \beta(1+r) = 1$. Thus, if the household is debt constrained, $a_3 = 0$. Furthermore, since $\phi > 0$, from KKT conditions,

$$d_3 = (1+r_m)d_2 - \frac{(1+r_m)^2}{2+r_m}d_2$$

and therefore

$$c_2 = (1+r)a_2 - \frac{(1+r_m)^2}{2+r_m}d_2 - e.$$

Setting $d_2 = 0$, solve $u'((1+r)a_2 - e^*) = \beta(1+r_m)u'(\omega_3)$ for e^*. If $e^* \geq \bar{e}$, then $\phi(e) = 0$ for every e in the support, so $E[\phi] = 0$ and, with $\lambda_2 = 0$, equation 48 gives $\theta_2 = (\beta(1+r_m) - 1)E[u'(c_2)] > 0$. Thus, the household sets $d_2 = 0$. Hence $d_2 > 0$ requires $e^* < \bar{e}$, in which case the debt constraint binds for all $e \in (e^*, \bar{e}]$. □

Lemma 2. *$a_3 > 0$ only if $d_3 = 0$. Similarly, $d_3 > 0$ only if $a_3 = 0$*

Proof. From equations 47 and 49

$$u'(c_2) = u'(c_3) + \lambda_3$$
$$\theta_3 + u'(c_2) = \beta(1+r_m)u'(c_3) + \phi$$

Since $a_3 > 0 \iff \lambda_3 = 0$ and $\beta(1+r_m) > \beta(1+r) = 1$, if $u'(c_2) = u'(c_3)$ then

$$\theta_3 = \beta(1+r_m)u'(c_3) - u'(c_2) > 0 \iff d_3 = 0.$$

Therefore $a_3 > 0 \iff d_3 = 0$.

Similarly, $d_3 > 0 \iff \phi > 0$ which implies

$$\phi = u'(c_2) - \beta(1+r_m)u'(c_3).$$

Since $\beta(1+r_m) > 1$, it must therefore be the case that

$$\lambda_3 = u'(c_2) - u'(c_3) > 0 \iff a_3 = 0.$$

Therefore if $d_3 > 0 \iff a_3 = 0$. □

Lemma 3. *There exists an interval in the second period over which optimal decisions yield $a_3 = d_3 = 0$.*

Proof. $a_3 = 0$ optimally whenever $u'(c_2) \geq u'(c_3)$. Similarly, $d_3 = 0$ whenever

$$u'(c_2) \leq \beta(1+r_m)u'(c_3).$$

Let $\varepsilon > 0$. Since $\beta(1+r_m) > \beta(1+r) > 1$ there exists a number $\varepsilon > 0$ however small such that

$$0 < u'(c_2) - u'(c_3) = \varepsilon < \beta(1+r_m)u'(c_3) - u'(c_2)$$

Therefore, there always exists an interval over which $a_3 = d_3 = 0$. □

Lemma 4. *c_2 is decreasing in e and $u'(c_2)$ is increasing in e.*

Proof. Since $c_2 = (1+r)a_2 - (1+r_m)d_2 - e - a_3 + d_3$, this depends on the optimal choices of a_3 and d_3. For a given value of e, at the optimum, if $a_3 > 0$ then $d_3 = 0$ by Lemma 2. Beginning with $a_3 > 0$, since this implies $\lambda_3 = 0$, we know that $c_2 = c_3$ and thus we can solve a_3 explicitly as

$$a_3(e) = \frac{(1+r)a_2 - (1+r_m)d_2 - e - \omega_3}{2+r}.$$

Thus, for $a_3 > 0$, we can write

$$\begin{aligned} c_2(e) &= (1+r)a_2 - (1+r_m)d_2 - \frac{(1+r)a_2 - (1+r_m)d_2 - \omega_3}{2+r} - e + \frac{e}{2+r} \\ \implies \frac{\partial c_2}{\partial e} &= -\frac{1+r}{2+r} < 0. \end{aligned}$$

Then, by Lemma 3, when both $a_3 = 0$ and $d_3 = 0$, c_3 is clearly decreasing in e since $c_2'(e) = -1$.

Finally, by Lemma 2, when $d_3 > 0$, $a_3 = 0$. When $d_3 = \bar{d}(d_2)$, since this is independent of e, c_2 is clearly decreasing in e. When $d_3 \in (0, \bar{d}(d_2))$, we know that

$$\begin{aligned} u'((1+r)a_2 - (1+r_m)d_2 - e + d_3) &= \beta(1+r_m)u'(\omega_3 - (1+r_m)d_3) \\ \implies u''(c_2)\frac{dd_3}{de} - u''(c_2) &= -\beta(1+r_m)^2 u''(c_3)\frac{dd_3}{de} \\ \implies \frac{dd_3}{de} &= \frac{u''(c_2)}{u''(c_2) + \beta(1+r_m)^2 u''(c_3)} < 1 \end{aligned}$$

Therefore, taking the derivative of c_2 with respect to e, we find

$$\frac{\partial c_2(e)}{\partial e} = -1 + \frac{dd_3}{de} < 0 \tag{107}$$

Since c_2 is decreasing in e across all possible future choices of a_3 and d_3, c_2 is decreasing in e. Finally, since $u''(c_2) < 0$, $u'(c_2)$ is increasing in e. □

Lemma 5. *ϕ is weakly increasing in e.*

Proof. Let $\phi = \max\{u'(c_2) - \beta(1+r_m)u'(c_3), 0\}$. Over the interval where $\phi = 0$, ϕ is clearly non-decreasing. Over the interval where $\phi > 0$, taking the total derivative we can write

$$d\phi = -u''(c_2)\left(1 - \frac{\partial d_3(e)}{\partial e}\right)de - \beta(1+r_m)^2 u''(c_2)\frac{\partial d_3(e)}{\partial e}de$$

By lemma 2, $1 > \frac{\partial d_3(e)}{\partial e} \geq 0$. Therefore, since $u'' < 0$

$$\frac{d\phi}{de} \geq 0.$$

□

Proposition 4. *[Existence of Co-holding Solutions with Consumer Credit] Co-holding exists in the model with consumer credit. However, households choose weakly lower levels of debt when they have access to credit.*

Proof. Suppose the same household makes choices under two regimes, with and without consumer credit. First, consider a household with endowments such that

$$u'((1+r)a_2^* - \bar{e}) = \beta(1+r_m)u'(\omega_3).$$

For this household, $a_2^* > 0$ otherwise $u'((1+r)a_2^* - \bar{e}) \to -\infty$ which is clearly a contradiction. Furthermore, since $d_3 = 0$ solves the first-order condition, $d_3 > 0$ would not make the household better off here and thus $\phi = 0$, implying $d_2 = 0$ is optimal. Since $r_b > r_m$, this also implies that

$$u'((1+r)a_2^* - \bar{e}) < \beta(1+r_m)u'(\omega_3).$$

This implies there must be a third period endowment $\omega_3' > \omega_3$ such that

$$u'((1+r)a_2^* - \bar{e}) > \beta(1+r_m)u'(\omega_3')$$
$$u'((1+r)a_2^* - \bar{e}) < \beta(1+r_b)u'(\omega_3')$$

In this case, $a_3 < 0$ could not be optimal as it would violate first-order conditions. However, the household would be better off with $d_3 > 0$. Therefore, by concavity of the utility function, there must exist some $d_2 > 0$ with $\underline{m} = \frac{(1+r_m)^2}{2+r_m} d_2$ such that

$$\Delta c_1 = (\omega_1 - a_2(d_2 > 0) + d_2) - (\omega_1 - a_2^*) = (a_2(d_2) - a_2^*) + d_2 > 0 \tag{108}$$
$$\Delta c_2 = ((1+r)a_2(d_2) - e - \underline{m}) - ((1+r)a_2^* - e) = (1+r)(a_2(d_2) - a_2^*) - \underline{m} > 0 \tag{109}$$
$$\Delta c_3 = \omega_3 - \underline{m} - \omega_3 = -\underline{m} < 0 \tag{110}$$

and the household is better off. Therefore, co-holding positive assets and debt is still an optimal solution depending on initial endowments with credit.

Now consider endowments such that absent the consumer credit regime, the household optimally co-holds $(a_2^*, d_2^* > 0)$. Let a_2' and d_2' denote the decisions these households would have made with consumer credit. By proposition 3, $\phi^* > 0$ for draws where $e \in (e^*, \bar{e}]$. In this range, the household makes minimum payments on debt. Under the consumer credit regime, suppose that $a_2^* = a_2'$ and $d_2^* = d_2'$. In this case, if the households were sufficiently

constrained, it could be the case that

$$u'((1+r)a_2^* - e - \underline{m}) > \beta(1+r_m)u'(\omega_3 - \underline{m})$$

which would imply $a_3' < 0$ is optimal. However, with $a_3' < 0$ possible for some initial endowments, in general we find

$$\begin{aligned} E[\phi'] &= E[u'((1+r)a_2^* - e - \underline{m} - a_3') - \beta(1+r_m)u'(\omega_3 + (1+r)a_3' - \underline{m})] \\ &\leq E[u'((1+r)a_2^* - e - \underline{m}) - \beta(1+r_m)u'(\omega_3 - \underline{m})] = E[\phi^*]. \end{aligned}$$

Thus, by first-order conditions in both regimes,

$$\begin{aligned} d_2' &= \left\{d_2 \in \mathbb{R}_+ | \theta_2 + \beta E[\phi']\bar{d}(d_2) + u'(c_1') = \beta(1+r_m)u'(c_2')\right\} \\ &\leq d_2^* = \left\{d_2 \in \mathbb{R}_+ | \theta_2 + \beta E[\phi']\bar{d}(d_2) + u'(c_1') = \beta(1+r_m)u'(c_2')\right\} \end{aligned}$$

□

Proposition 5. *[No Arbitrage Between Debt and Viscous Assets] There is no arbitrage opportunity when* $0 < r < r_m < r_A$ *if and only if*

$$(1+r_A)^2 \leq \frac{(1+r)(1+r_m)^2}{(2+r_m)(1+r) - (1+r_m)^2}$$

Proof. Arbitrage opportunities will exist if it is possible to increase debt in order to increase consumption in a given period without decreasing consumption in any other period. Suppose prior to a loan market opening a household optimizes by choosing c_1^*, c_2^*, c_3^*, a_2^*, and A_2^*. After these choices are made, the loan market opens and we allow households to change their choices. We'll denote new choices with the prime notation (e.g., c_1'). There is an arbitrage in the model if and only if

$$\begin{aligned} \Delta c_1 &= c_1' - c_1^* = 1 - \Delta a_2 - \Delta A_2 \geq 0 \\ \Delta c_2 &= c_2' - c_2^* = (1+r)\Delta a_2 + (1 + (1-\kappa)r_A)(\Delta A_2 - \Delta A_3) - m_2 \geq 0 \\ \Delta c_3 &= c_3' - c_3^* = (1+r)\Delta a_3 + (1+r_A)^2 \Delta A_2 - (1+r_m)((1+r_m) - m_2) \geq 0. \end{aligned}$$

Since argument is just based on arbitrage, $\Delta A_3 = \Delta A_2$ and $\Delta a_3 = 0$ because arbitrage implies that a_3^* and A_3^* is still attainable across all states. The term $(1+r_m) - m_2$ appears because $d_3 = (1+r_m) - m_2$. If the household makes a maximum second period payment on

debt without decreasing period 2 consumption, then

$$m_2 = (1+r)\Delta a_2.$$

Setting $\Delta a_2 = 1 - \Delta A_2$ we get

$$\Delta A_2 \geq \frac{(1+r_m)(r_m - r)}{(1+r_A)^2 - (1+r_m)(1+r)} = L.$$

This gives us a natural lower bound. If it is not possible to choose $\Delta A_2 \geq L$ then there is no arbitrage. Now, since there is a minimum second period payment on debt

$$\Delta c_2 = (1+r)(1-\Delta A_2) - \underline{m}_2 \geq 0$$

which simplifies to

$$\Delta A_2 \leq 1 - \frac{(1+r_m)^2}{(2+r_m)(1+r)} = U.$$

If $U \leq L$ then there is no arbitrage. Algebraically, this simplifies to

$$(1+r_A)^2 \leq \frac{(1+r)(1+r_m)^2}{(2+r_m)(1+r) - (1+r_m)^2} \tag{111}$$

Therefore, if 111 holds, there is no arbitrage. □

Proposition 6. *[Three-way co-holding solutions] Suppose the bound in Proposition 5 holds strictly and the viscous asset is held to period 3.*

1. *Without expense risk, no household holds $a_2 > 0$, $A_2 > 0$ and $d_2 > 0$ simultaneously.*
2. *With expense risk, a solution with $a_2 > 0$, $A_2 > 0$ and $d_2 > 0$ requires that the household be debt constrained in period 2 with probability strictly between zero and one.*

Proof. With $\beta(1+r) = 1$, let $D = (2+r_m)(1+r) - (1+r_m)^2$ and

$$\Lambda = \frac{\beta^2(1+r_m)^2}{\beta(1+r_m) - (\beta(1+r_m)-1)(2+r_m)}$$
$$= \frac{\beta^2(1+r_m)^2}{\big((2+r_m) - \beta(1+r_m)^2\big)} = \frac{\beta(1+r_m)^2}{D}. \tag{112}$$

Similarly, multiplying the bound in Proposition 5 by β^2 gives $\beta^2(1+r_A)^2 \leq \beta(1+r_m)^2/D = \Lambda$.

At a solution with $a_2 > 0$, $A_2 > 0$ (held to period 3) and $d_2 > 0$, the first-order conditions

for a_2 and A_2 hold with equality:

$$u'(c_1) = E[u'(c_2)], \qquad u'(c_1) = \beta^2(1+r_A)^2 E[u'(c_3)],$$

so $E[u'(c_2)]/E[u'(c_3)] = \beta^2(1+r_A)^2$. From the first-order conditions

$$\bar{d}'(d_2)\, E[\phi] = (r_m - r)\, E[u'(c_2)], \text{ where } \phi = \left[u'(c_2(e)) - \beta(1+r_m)u'(c_3(e))\right]^+.$$

If the household is debt constrained in no state, $E[\phi] = 0$ and the condition cannot hold, so $d_2 = 0$. If it is debt constrained in every state, then $E[\phi] = E[u'(c_2)] - \beta(1+r_m)E[u'(c_3)]$, and the condition rearranges to

$$\frac{E[u'(c_2)]}{E[u'(c_3)]} = \frac{\beta^2(1+r_m)^2}{\beta(1+r_m) - (\beta(1+r_m)-1)(2+r_m)} = \Lambda,$$

which would contradict $E[u'(c_2)]/E[u'(c_3)] = \beta^2(1+r_A)^2 < \Lambda$. Hence the household must be debt constrained in some states and not in others. □